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On eigenvalues of random d-regular graphs

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1 Introduction

The famous "singularity" problem for random matrices is the determination of the probability of the adjacency matrix of a random graph being singular for various distribution : the Erdős–Rényi model, the uniform law on d-regular matrices, the configuration model, etc.

1.1 The d-regular configuration model

The configuration model of d-regular directed graphs introduced by Bollobás [3], generates a random d-regular graph by the following procedure:

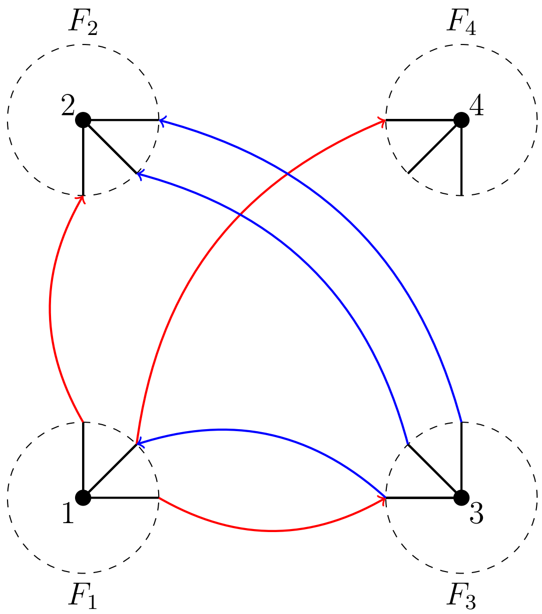

Figure 1: Configuration model, 3-regular graph.

1. To each vertex $i \in [\![1, n]\!]$ we associate a fiber $F_i = \{i_1, \cdots, i_d\}$, such so there are nd points in total.
2. Select at random permutation $\mathcal{P}$ on $F = \cup_{i \in [\![1,n]\!]} F_i$ uniformly. For $i, j \in [\![1, n]\!]$ we connect them for each $i_k \in F_i$, such that $P(i_k) \in F_j$.

The resulting random graph $M_{n,d}$ is a d-regular directed multi-graph. We see that $|M_{n,d}| = (nd)!$. For the undirected configuration model we take $2|dn$, and we follow a similar procedure:

1. To each vertex $i \in [\![1, n]\!]$ we associate a fiber $F_i = \{i_1, \cdots, i_d\}$, such so there are nd points in total.
2. Select a pairing $\mathcal{P}$ on $F = \cup_{i \in [\![1,n]\!]} F_i$ uniformly and add an edge $k' - l'$, if $\{k', l'\} \in \mathcal{P}$.

The resulting random graph $G_{n,d}$ is a d-regular undirected multi-graph. To create a pairing we repeatedly select 2 vertices that have not been previously selected and match them together, so the number of ordered pairing is $\binom{nd}{2}\binom{nd-2}{2}\cdots\binom{2}{2} = 2^{-nd/2}(nd)!$. Finally to get unordered pairing, we omit the ordering, i.e. :

$$|G_{n,d}| = \frac{1}{(nd/2)!}\prod_{i=0}^{nd/2}\binom{nd-2i}{2} = \frac{(nd)!}{2^{nd/2}(nd/2)!}$$

1.2 Invertibility of adjacency matrix of random d-regular matrices

This problem was solved for the d-regular configuration model with fixed d by Huang [7]. In the random d-regular graph model we lose the independence of the vertices we get for example in the Erdős–Rényi model, which poses significant issues for the singularity problem. In the paper, Huang proved that the random d-regular matrices are non-singular with high probability. Instead of studying the singularity problem in $\mathbb{R}$, the key idea is to embed the matrices in $\mathbb{F}_p$. A matrix is singular in $\mathbb{F}_p$ if $\det \in p\mathbb{Z}$, so one might expect that matrices are singular with positive probability, however the use of arithmetic structures in $\mathbb{F}_p$ gives better estimates of the singularity probability. Precisely, he showed with $p \ll n^{-\mathfrak{d}}$, ($\mathfrak{d}$ depending only on d), that :

$$\mathbb{P}(A(G) \text{ is singular in } \mathbb{F}_p) \leq \frac{1 + \underset{n\to\infty}{o}(1)}{p-1} \tag{1.1}$$

where $A(G)$ is the adjacency matrix of G. Deriving the following theorem :

Theorem 1.1. *Let $d \geq 3$ be a fixed integer. There exits $\mathfrak{d} > 0$, as n goes to infinity :*

$$\mathbb{P}(A(G) \text{ is singular in } \mathbb{R}) = o(n^{-\mathfrak{d}})$$

for G following the d-regular configuration model on graphs with n vertices.

The proof transforms the problem of counting :

$$|\{(w,G)|A(G)w = 0\}| \tag{1.2}$$

to a random walk in $\mathbb{Z}^p$, then separating cases and studying them using a local limit theorem estimate an a large deviation estimate accordingly. By refining the separation in these categories we managed to simplify the proof, and to show that $\mathfrak{d}$ can be taken arbitrarily close to 1 independently of d.

1.3 Extension to other eigenvalues

We then tried to generalize the method developed for the eigenvalue 0 and study the probability of $\lambda \in \mathrm{Sp}(G)$ for a fixed λ. Fix $\lambda \in \mathfrak{A}$ and P its minimal monic polynomial, let h be its degree. Let $A \in M_n(\mathbb{Z})$, if λ is an eigenvalue of A then $P(A)$ is singular in $\mathbb{R}^n$, therefore in each $\mathbb{F}_p^n$, i.e. there is a non-zero vector $w \in \mathbb{F}_p^n$ such that $P(A)w = 0$. In fact there are at least $p-1$ vectors $w_k = kw$ for $k \neq 0 \in \mathbb{F}_p$. Therefore we have :

$$(p-1)|\{G|\lambda \in \mathrm{Sp}(G)\}| \leq |\{(w,G)|P(A(G))w = 0\}| \tag{1.3}$$

To study this factor we notice that the factor $(A^l w)_i$ counts the number of paths starting in i of length l given a mass w_j if they end in j. Thus $P(A)w$ is closely related to the h-neighborhood of vertices in G.

To count $|\{(w,G)|P(A(G))w=0\}|$, first we see w as a coloring of G in $\mathbb{F}_p$ ($i \in V$ is colored w_i), we will proceed in several steps :

- Given a colored graph, we will use the generalized configuration introduced by Bordenave Caputo [4] to determine the number of multi-graphs that have the same colored h-neighborhood.
- We will then generalize the results of Bordenave Coste [5] and give a condition on a list of n unlabeled h-neighborhood colored by $\mathbb{F}_p$ so that the exists a simple graph with such h-neighborhood.
- Finally now that knowing the h-neighborhood of the graph, we will give a simple condition such that we have $P(A(G))w=0$.

Part I
Probability of being non-singular

2 Random Walk Interpretation

In this section, we enumerate $\{G \in M_{n,d}|A(G)v = 0 \in \mathbb{F}_p\}$ as the number of certain walk paths, and then transforming that number into a random walk. We then give an exponential bound on the Fourier transform of the walk where the transform is small.

2.1 Notations

We introduce some notations, let $\Phi : \cup_{k\geq 1}\mathbb{F}_p^k \to \mathbb{N}^p$ be the counting function :

$$\forall k \in \mathbb{N}^*,\ \Phi(a_1,\cdots,a_k) = \left(\sum_{i=1}^k \mathbb{1}_{a_i=0},\cdots,\sum_{i=1}^k \mathbb{1}_{a_i=p-1}\right)$$

If $\sum_{k=1}^{p-1} a_k = n$, then we define the sphere $\mathcal{S}^n(a_0,\cdots,a_{p-1}) \subset \mathbb{F}_p^n$ as :

$$\mathcal{S}^n(a_0,\cdots,a_{p-1}) = \{v \in \mathbb{F}_p^n|\Phi(v) = (a_0,\cdots,a_{p-1})\}$$

The cardinalty of $\mathcal{S}^n(a_0,\cdots,a_{p-1})$ is the multinomial :

$$|\mathcal{S}^n(a_0,\cdots,a_{p-1})| = \binom{n}{a_0,\cdots,a_{p-1}} = \frac{n!}{\prod_{i\leq p-1} a_i!}$$

And $\mathbb{F}_p^n$ can be decomposed as :

$$\mathbb{F}_p^n = \bigsqcup_{a_0+\cdots+a_{p-1}=n} \mathcal{S}^n(a_0,\cdots,a_{p-1})$$

We denote $\mathbb{F}_{d,p}^0$, the zero sum vectors of $\mathbb{F}_p^d$. We introduce the multiset $\mathcal{U}_{d,p}$:

$$\begin{aligned}\mathcal{U}_{d,p} &= \{\Phi(a_0, \cdots, a_{d-1}) : a_0 + \cdots, +a_{d-1} = 0\} \\ &= \left\{\Phi\left(a_0, \cdots, a_{d-2}, -\sum_{k=0}^{d-2} a_k\right) : \mathbf{a} = \left(a_0, \cdots, a_{d-2}, -\sum_{k=0}^{d-2} a_k\right) \in \mathbb{F}_{d,p}^0\right\}\end{aligned}$$

As a multiset $|\mathcal{U}_{d,p}| = |\mathbb{F}_{d,p}^0| = p^{d-1}$.

2.2 Preliminary results

Proposition 2.1. *Let $d \geq 3$ be a fixed integer, and a prime number p. Fix $v \in \mathcal{S}^n(a_0, \cdots, a_{p-1})$, we have :*

$$\begin{aligned}|\{\mathcal{G} \in M_{n,d} | A(\mathcal{G})v = 0\}| &= \prod_{j=0}^{p-1}(da_k)! \left|\left\{(\boldsymbol{u}_k)_{k\leq n} \in \mathcal{U}_{d,p}^n | \sum_{k=1}^{n} \boldsymbol{u}_k = d(a_0, \cdots, a_{p-1})\right\}\right| \\ &= \prod_{j=0}^{p-1}(da_k)! p^{n(d-1)} \mathbb{P}(X_1 + \cdots + X_n = d(a_0, \cdots, a_{p-1}))\end{aligned}$$

where $X_1, \cdots, X_n$ are independent uniform distributions over $\mathcal{U}_{d,p}$.

Proof. We introduce the equivalence relation on $F = \bigcup_{k=1}^{n} F_k$:

$$k' \sim l' \iff v_k = v_l$$

and $\pi : k_i \mapsto v_k$ the projection on $\mathbb{F}_p$. Then for each $j \in \mathbb{F}_p$:

$$|\pi^{-1}(j)| = \sum_{v_l = j} |F_l| = da_j$$

For a permutation $\mathcal{P}$ of nd points, we associate the map $f_{\mathcal{P}}$ that colors fibers :

$$f_{\mathcal{P}} : \begin{cases} F & \to \mathbb{F}_p \\ k' \in F_k & \mapsto v_l \text{ such that } \mathcal{P}(k') \in F_l \end{cases}$$

then $f_{\mathcal{P}} = \pi \circ \mathcal{P}$. For each $j \in \mathbb{F}_p$:

$$|f_{\mathcal{P}}^{-1}(j)| = |\mathcal{P}^{-1}(\pi^{-1}(j))| = |\pi^{-1}(j)| = da_j \tag{2.1}$$

On the contrary, if a given map $f : F \to \mathbb{F}_p$ verifies :

$$\forall j \in \mathbb{F}_p, |f^{-1}(j)| = da_j = |\pi^{-1}(j)|$$

then f derives from a permutation. Indeed for each permutation $\mathcal{P}$ that pairs elements of $f^{-1}(j)$ with $\pi^{-1}(j)$, verifies $f = f_{\mathcal{P}}$, and these are the only ones. Therefore they are exactly $\prod_{j \in \mathbb{F}_p}(da_j)!$ permutations $\mathcal{P}$ such that $f = f_{\mathcal{P}}$

Let $G \in M_{n,d}$ corresponding to a permutation $\mathcal{P}$. $A(G)v = 0$ iff :

$$\forall k \in [\![1, n]\!], \quad \sum_{l=1}^{n} a_{k,l} v_l = 0 \tag{2.2}$$

As :

$$a_{k,l} = \sum_{k' \in F_k} \mathbb{1}_{\{\mathcal{P}(k') \in F_l\}}$$

Then :

$$\begin{aligned}\sum_{l=1}^{n} a_{k,l} v_l &= \sum_{k' \in F_k} \sum_{l=1}^{n} \mathbb{1}_{\{\mathcal{P}(k') \in F_l\}} v_l \\ &= \sum_{k' \in F_k} f_{\mathcal{P}}(k')\end{aligned} \tag{2.3}$$

therefore $A(\mathcal{G})v = 0 \iff \forall k \in [\![1, n]\!],\ \{\Phi(f(k')), k' \in F_k\} \in \mathcal{U}_{d,p}$. And the number of maps f that verify this condition and (2.1) are:

$$\begin{aligned}&\{f |\ ((f(1_i)_{i\le d}), \cdots, (f(n_i)_{i\le d}) \in \mathcal{U}_{d,p}^n,\ |f_{\mathcal{P}}^{-1}(j)| = da_j\} \\ &= \left\{ f |\ ((f(1_i)_{i\le d}), \cdots, (f(n_i)_{i\le d}) \in \mathcal{U}_{d,p}^n,\ \sum_{k=1}^{n} \underbrace{\sum_{k' \in F_k} \mathbb{1}_{f(k')=j}}_{=\Phi((f(k_i)_{i\le d})_j} = da_j \right\} \\ &\simeq \left\{ (\boldsymbol{u}_k)_{k\le n} \in \mathcal{U}_{d,p}^n |\ \sum_{k=1}^{n} \boldsymbol{u}_k = d(a_0, \cdots, a_{p-1}) \right\}\end{aligned}$$

And for each of these maps they are $\prod_{j\in\mathbb{F}_p}(da_j)!$ permutations associated, therefore :

$$|\{\mathcal{G} \in M_{n,d} | A(\mathcal{G})v = 0\}| = \prod_{j=0}^{p-1}(da_k)! \left| \left\{ (\boldsymbol{u}_k)_{k\le n} \in \mathcal{U}_{d,p}^n |\ \sum_{k=1}^{n} \boldsymbol{u}_k = d(a_0, \cdots, a_{p-1}) \right\} \right|$$

□

2.3 Fourier transform bound

Let X be a random vector uniform distributed over $\mathcal{U}_{d,p}$. Then mean of X is given by:

$$\begin{aligned}\mathbb{E}[X_j] &= \frac{1}{p^{d-1}} \sum_{\mathbf{a} \in \mathbb{F}_{d,p}^0} \sum_{k=1}^{d} \mathbb{1}_{a_k=j} = \frac{1}{p^{d-1}} \sum_{k=1}^{d} \sum_{\mathbf{a} \in \mathbb{F}_{d,p}^0} \mathbb{1}_{a_k=j} \\ &= \frac{1}{p^{d-1}} \sum_{k=1}^{d} \underbrace{\sum_{a_1+\cdots+a_{d-1}+j=0}}_{=p^{d-2}} \\ &= \frac{1}{p} \sum_{k=1}^{d} = \frac{d}{p}\end{aligned} \tag{2.4}$$

The covariance of X if given by :

$$
\begin{aligned}
\mathbb{E}[(X_i - d/p)(X_j - d/p)] &= \mathbb{E}[X_i X_j] - 2\frac{d^2}{p^2} + \frac{d^2}{p^2} \\
&= \frac{1}{p^{d-1}} \sum_{\mathbf{a}\in\mathbb{F}^0_{d,p}} \Phi_i(\mathbf{a})\Phi_j(\mathbf{a}) - \frac{d^2}{p^2} \\
&= \frac{1}{p^{d-1}} \sum_{\mathbf{a}\in\mathbb{F}^0_{d,p}} \sum_{k,l\leq d} \mathbb{1}_{a_k=i}\mathbb{1}_{a_l=j} - \frac{d^2}{p^2} \\
&= \frac{1}{p^{d-1}} \sum_{\mathbf{a}\in\mathbb{F}^0_{d,p}} \delta_{ij} \sum_{k=1}^{d} \mathbb{1}_{a_k=i} \qquad (2.5)\\
&+ \sum_{k\neq l} \mathbb{1}_{a_k=i}\mathbb{1}_{a_l=j} - \frac{d^2}{p^2} \\
&= \delta_{ij}\frac{d}{p} - \frac{d^2}{p^2} + \frac{1}{p^{d-1}} \sum_{k\neq l} \sum_{a_1+\cdots+a_{d-2}+i+j=0} \\
&= \delta_{ij}\frac{d}{p} - \frac{d^2}{p^2} + \frac{d^2-d}{p^2} \\
&= \delta_{ij}\frac{d}{p} - \frac{d}{p^2}
\end{aligned}
$$

We denote:

$$\mu = \mathbb{E}[X], \quad \Sigma = \mathbb{E}[(X-\mu)(X-\mu)^t] = \frac{d}{p}I_p - \frac{d}{p^2}\mathbf{1}_p$$

And the characteristic function of X as :

$$\phi_X(\mathbf{t}) = \mathbb{E}[e^{i<\mathbf{t},X>}]$$

Because X is uniform $|\phi(\boldsymbol{t})| = 1$ iff all the exponential have the same direction, i.e. :

$$\forall \boldsymbol{a} \in \mathbb{F}^0_{d,p}, \quad < t, \Phi(\boldsymbol{a}) > \equiv < t, \Phi(0) > = dt_1 \mod 2\pi \qquad (2.6)$$

this conditions is stable by sum. For any $\boldsymbol{a} \in \mathbb{F}^d_p$ we have :

$$
\begin{aligned}
\sum_{k=1}^{p} \Phi(\boldsymbol{a})_k &= \sum_{k=0}^{p-1}\sum_{i=1}^{d} \mathbb{1}_{a_i=k} \\
&= \sum_{i=1}^{d} \underbrace{\sum_{k=0}^{p-1} \mathbb{1}_{a_i=k}}_{=1} = d
\end{aligned}
$$

thus $(1,\cdots,1)\mathbb{R}$ verifies the condition (2.6).

Let $t \in \mathbb{R}^p$ verify condition (2.6), without loss of generality (by replacing t with $t - (t_0, \cdots, t_0)$) we can assume $t_0 = 0$. $\Phi(k, p-k, 0\cdots) = (\cdots 0, \underbrace{1}_{\text{position } k}, 0\cdots, 0, \underbrace{1}_{\text{position } p-k}, 0\cdots)$, therefore :

$$\forall k \in \mathbb{F}_p,\ t_k + t_{p-k} \equiv 0 \mod 2\pi \tag{2.7}$$

$(1, k-1, p-k, 0\cdots) \in \mathbb{F}^0_{d,p}$, therefore :

$$\forall k \in \mathbb{F}_p,\ t_1 + t_k + t_{p-k} \equiv 0 \mod 2\pi \tag{2.8}$$

By subtracting (2.7) and (2.8) we get :

$$\forall k \in \mathbb{F}_p,\ t_k \equiv t_1 + t_{k-1} \mod 2\pi \tag{2.9}$$

Thus, $\forall k \in \mathbb{F}_p$, $t_k \equiv kt_1 \mod 2\pi$, especially, $t_1 \equiv (p+1)t_1 \mod 2\pi$. So we write t_1 as $2n\pi/p$ and t_k as $k2n\pi/p + 2m_k\pi$. We define $\alpha_p = (0, 1/p, \cdots, p-1/p)$, then we have proven that :

$$\{\boldsymbol{t} \in \mathbb{R}^p | |\phi_X(\boldsymbol{t})| = 1\} \subset 2\pi\mathbb{Z}\alpha_p + \mathbb{R}\mathbf{1}_p + 2\pi\mathbb{Z}^p$$

On the contrary, if $\boldsymbol{t} \equiv n2\pi\alpha_p + \lambda\mathbf{1} \mod 2\pi$, then for $\boldsymbol{a} \in \mathbb{F}^0_{d,p}$ we have $\sum_{i=1}^d a_i \in p\mathbb{Z}$, therefore :

$$\begin{aligned} < \boldsymbol{t}, \Phi(\boldsymbol{a}) > &\equiv \sum_{k=0}^{p-1} \left(\frac{nk2\pi}{p} + \lambda\right) \sum_{i=1}^{d} \mathbb{1}_{a_i = k} \mod 2\pi \\ &\equiv \sum_{i=1}^{d} \left(\frac{na_i 2\pi}{p} + \lambda\right) \mod 2\pi \\ &\equiv d\lambda \mod 2\pi \end{aligned}$$

We have proven :

Lemma 1.

$$|\phi_X(\boldsymbol{t})| = 1 \iff \boldsymbol{t} \in 2\pi\mathbb{Z}\alpha_p + \mathbb{R}\boldsymbol{1}_p + 2\pi\mathbb{Z}^p$$

Proposition 2.2. *For any $\delta > 0$ small enough, and $t \in (2\pi\mathbb{R}^p/\mathbb{Z}^p)/\cup_{j=0}^p B_j(\delta)$, there exists a constant $c(d) > 0$, that depends only on d, such as :*

$$|\phi_{X-\mu}(\boldsymbol{t})| \leq 1 - pc(d)\delta^2$$

where $B_j(\delta) = \{\boldsymbol{x} \in \mathbb{R}^p | \ \|\boldsymbol{x} - 2j\pi\alpha_p\|_\infty \leq \delta\}$

Proof. Let $c(d) = 1/(8 \times 40^2 d^3)$. By contradiction, we assume there is some $\boldsymbol{t}$:

$$\left| \frac{1}{p^{d-1}} \sum_{\omega \in \mathcal{U}_{d,p}} e^{i<\boldsymbol{t},\omega>} \right| \geq 1 - pc(\delta)\delta^2$$

As before, by shifting $\boldsymbol{t}$ we can assume $t_0 = 0$. We denote $\psi = \arg \phi_X(\boldsymbol{t})$. Let $\epsilon = 2\delta\sqrt{2dc(d)}$. Then :

$$1 - \cos(\epsilon) \geq \epsilon^2/4 = 2dc(d)\delta^2 \tag{2.10}$$

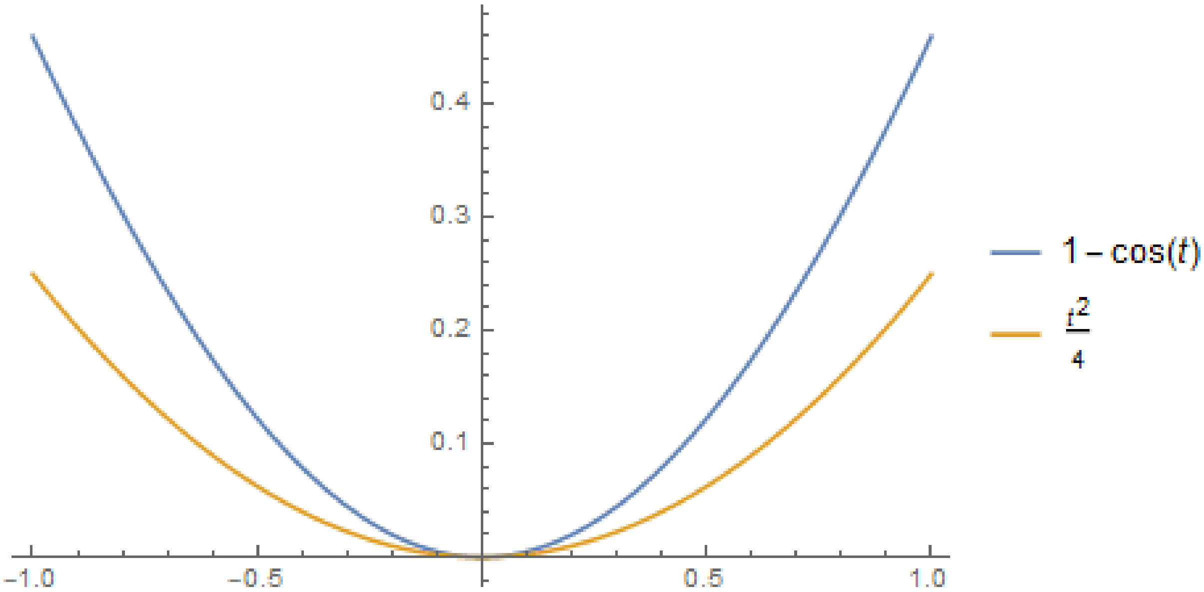

Figure 2: Lower bound of $1 - \cos$

We define the set of non-equidistributed :

$$\mathcal{U}' = \{\omega \in \mathcal{U}_{d,p} |\ |e_\omega =< \boldsymbol{t}, \omega > -\psi - 2\pi n_\omega| > \epsilon\}$$

where n_ω is such that $| < \boldsymbol{t}, \omega > -2\pi n_\omega| < \pi$. Then $|\mathcal{U}'| \leq p^{d-2}/(2d)$, otherwise :

$$\begin{aligned}
\left| \frac{1}{p^{d-1}} \sum_{\omega \in \mathcal{U}_{d,p}} e^{i<\boldsymbol{t},\omega>} \right| &= \frac{1}{p^{d-1}} \Re \left| \sum_{\omega \in \mathcal{U}_{d,p}} e^{i<\boldsymbol{t},\omega>} \right| \\
&= \frac{1}{p^{d-1}} \Re \left(\sum_{\omega \in \mathcal{U}_{d,p}} e^{i<\boldsymbol{t},\omega>} \right) e^{-i\psi} \\
&= \frac{1}{p^{d-1}} \Re \sum_{\omega \in \mathcal{U}_{d,p}} e^{ie_\omega} \\
&< \frac{1}{p^{d-1}} (|\mathcal{U}_{d,p}/\mathcal{U}'| + \sum_{\omega \in \mathcal{U}'} \cos(\epsilon) \\
&< 1 + \frac{|\mathcal{U}'|}{p^{d-1}} (-1 + 1 - 2dc(d)\delta^2) \\
&< 1 - pc(\delta)\delta^2
\end{aligned}$$

We consider zero sum $d \times d$ array in $\mathbb{F}_p$. Fix $\boldsymbol{a}^1 = (a_1^1, \cdots, a_d^1) \in \mathbb{F}_{d,p}^0$, the total number of zero sum array with first row given by a^1 is $p^{(d-1)(d-2)}$ (given a $d-1 \times d-1$ array there is one and only one way to construct a $d \times d$ zero sum array from it). For any $\boldsymbol{b} \in \mathbb{F}_{d,p}^0$, the total number of zero arrays with first row $\boldsymbol{a}^1$ and one other row or column given by $\boldsymbol{b}$ is at most :

$$\underbrace{(d-1)p^{(d-1)(d-3)}}_{b \text{ is a row}} + \underbrace{dp^{(d-1)(d-3)}}_{b \text{ is a column}} \tag{2.11}$$

We also have :

$$\begin{aligned}|\mathcal{U}'|((d-1)p^{(d-1)(d-3)}+dp^{(d-2)(d-2)}) &\leq \frac{1}{2}(p^{(d-2)(d-2)+d-2}+p^{(d-1)(d-3)+d-2})\\ &\leq \frac{1}{2}(p^{(d-1)(d-2)}+p^{(d-1)(d-2)-1})\\ &< p^{(d-1)(d-2)}\end{aligned}$$

If, for all zero sum matrix with rows a^i (first row $\mathbf{a}^1$ fixed before), column b^j they were at least a row or a column such that $\Phi(a^i) \in \mathcal{U}'$ or $\Phi(b^j) \in \mathcal{U}'$, then we would have :

$$\begin{aligned}p^{(d-1)(d-2)} = &\sum_{\phi(\boldsymbol{b})\in\mathcal{U}_{d,p}} |\{\text{zero sum matrix with } \boldsymbol{b} \text{ as a column or row, and}\\ &\boldsymbol{a}^1 \text{ as the first row}\}|\\ &\leq |\mathcal{U}'|((d-1)p^{(d-1)(d-3)}+dp^{(d-2)(d-2)}) < p^{(d-1)(d-2)}\end{aligned}$$

thus there is at least a zero sum matrix with first row a^1, such that :

$$\forall i \geq 2, \Phi(a^i) \notin \mathcal{U}' \quad \forall j \geq 1, \Phi(b^j) \notin \mathcal{U}'$$

We have $a^i_j = b^j_i$, therefore for all $k \in \mathbb{F}_p$:

$$\sum_{i=1}^{d} \Phi_k(\boldsymbol{a}^i) = \sum_{i=1}^{d}\sum_{j=1}^{d} \mathbb{1}_{a^i_j=k} = \sum_{i=1}^{d}\sum_{j=1}^{d} \mathbb{1}_{b^j_i=k} = \sum_{j=1}^{d} \Phi_k(\boldsymbol{b}^j)$$

And :

$$\begin{aligned}< \boldsymbol{t}, \Phi(\boldsymbol{a}^1) > &= \sum_{j=1}^{d} < \boldsymbol{t}, \Phi_k(\boldsymbol{b}^j) > - \sum_{i=2}^{d} < \boldsymbol{t}, \Phi(\boldsymbol{a}^i) >\\ &= 2\pi n_{\Phi(\boldsymbol{a}^1)} + \psi + \underbrace{\sum_{j=1}^{d} \epsilon_{\Phi_k(\boldsymbol{b}^j)} - \sum_{i=2}^{d} \epsilon_{\Phi(\boldsymbol{a}^i)}}_{=\epsilon_{\Phi(a_1)}}\end{aligned}$$

Thus, uniformly, $|\epsilon_w| \leq 2d\epsilon$. Especially, if $\boldsymbol{a}^1 = (0, \cdots, 0)$ we get $|\psi| \leq 2d\epsilon$, thus uniformly on $\mathcal{U}_{d,p}$ we have :

$$\forall \omega \in \mathcal{U}_{d,p}, \ | < \boldsymbol{t}, \omega > | \leq 4d\epsilon \pmod{2\pi} \tag{2.12}$$

We proceed similarly to 2.3 and considering two family of vectors $u_k = (d-2)e_0 + e_k + e_{p-k}$, $v_k = (d-3)e_0 + e_1 + e_{k-1} + e_{p-k}$, we get :

$$\forall k \in \mathbb{F}_p, \ t_1 \equiv kt_k + \sum_{j=2}^{k}(e_{v_j} - e_{u_j}) \mod 2\pi \tag{2.13}$$

We can shift $\boldsymbol{t}$ in $2\pi\mathbb{Z}^p$ and $2\pi\alpha_p\mathbb{Z}$ and assume there is an equality in (2.13) and take $|t_1| \leq 1/2p$. Thanks to the bound (2.12) we have $|t_k| \leq k/2p + 8d\epsilon(k-1) \leq 1$, for δ small enough.

Let $k_{\max} = \text{argmax}_k t_k$ and $k_{\min} = \text{argmin}_k t_k$. By taking $u = u_{k_{\max}}$ and $v = (d-3)e_0 + e_{k_{\min}} + e_{k_{\max}-k_{\min}} + e_{p-k_{\max}}$ in (2.12), we get :

$$3 \geq |t_{k_{\max}} - t_{k_{\min}} - t_{k_{\max}-k_{\min}}| \geq 2\pi|n_u - n_v| - |\epsilon_u - \epsilon_v|$$

therefore $n_u = n_v$, and :

$$t_{k_{\max}} = t_{k_{\min}} + t_{k_{\max}-k_{\min}} + \epsilon_1$$

where $|\epsilon_1 = \epsilon_u - \epsilon_v| \leq 8d\epsilon$, by symmetry we also have $t_{k_{\min}} = t_{k_{\max}} + t_{k_{\min}-k_{\max}} + \epsilon_2$ with $|\epsilon_2| \leq 8d\epsilon$. By subtracting both equations we get :

$$\begin{aligned} 2(t_{k_{\max}} - t_{k_{\min}}) &= t_{k_{\max}-k_{\min}} - t_{k_{\min}-k_{\max}} + \epsilon_1 - \epsilon_2 \\ &\leq t_{k_{\max}} - t_{k_{\min}} + \epsilon_1 - \epsilon_2 \\ |t_{k_{\max}} - t_{k_{\min}}| &\leq |\epsilon_1| + |\epsilon_2| \leq 16d\epsilon \end{aligned}$$

Furthermore :

$$|t_2 - t_1| = |t_1 + e_{v_1} - e_{u_1}| \leq |t_{k_{\max}} - t_{k_{\min}}| \leq 16d\epsilon$$

So $|t_1| \leq 24d\epsilon/p$ and finally, for any k :

$$|t_k| \leq |t_k - t_1| + |t_1| \leq |t_{k_{\max}} - t_{k_{\min}}| + |t_1| \leq 40d\epsilon$$

We recall, the definition of ϵ and $c(d)$, and we get :

$$40d\epsilon = 40d \times 2\delta\sqrt{\frac{2d}{8 \times 40^2 d^3}} = \delta$$

□

3 Proof of Theorem 1.1 for directed graphs

Thanks to proposition 2.1, we can rewrite the theorem as :

$$\begin{aligned} &\sum_{\substack{a_0+\cdots+a_{p-1}=n \\ a_0 \neq n}} \binom{n}{a_0, \cdots, a_{p-1}} \binom{dn}{da_0, \cdots, da_{p-1}}^{-1} \\ &\times p^{n(d-1)} \mathbb{P}(S_n = X_1 + \cdots + X_n = (da_0, \cdots, da_{p-1})) = 1 + o(1) \end{aligned} \tag{3.1}$$

We decompose p-tuples $(a_0, \cdots, a_{p-1})$ into two classes :

- (Equidistributed) $\mathcal{E}$ is the set of p-tuples $(a_0, \cdots, a_{p-1})$, such that :

$$\max_{0 \leq j \leq p-1} \left| \frac{a_j}{n} - \frac{1}{p} \right| \leq \sqrt{\delta(n)}/p \tag{3.2}$$

  where $\delta \to 0$. Then we also have $\|\boldsymbol{a}/n - 1/p\|^2 \leq \delta/p$. In the article by Huang this bound was $\ln(n)/n$, but we will see later that we can take $\delta = \ln(n)^{2/3}/n^{1/3}$.

- (Non-Equidistributed) $\mathcal{N}$, the others.

3.1 Local limit theorem estimate

In this section, we estimate the sum of terms in (3.1) corresponding to equidistributed p-tuples, using a local limit theorem.

Proposition 3.1. *Let $d \geq 3$ be a fixed integer, p a prime such that* $\gcd(p, d) = 1$. *Then for n sufficiently large :*

$$\sum_{\boldsymbol{a} \in \mathcal{E}} \sum_{\boldsymbol{v} \in \mathcal{S}^n(a_0, \cdots, a_{p-1})} |\{\mathcal{G} | A(\mathcal{G})v = 0\}| \leq (1 + o(1))|M_{n,d}| \tag{3.3}$$

Proof. We first estimate the factor on the right hand side of (3.1), using the Stirling formula :

$$\begin{aligned} \frac{(dn)!}{n!} &\sim \sqrt{\frac{2\pi dn}{2\pi n}} \left(\frac{dn}{e}\right)^{dn} \left(\frac{e}{n}\right)^n \\ &= \sqrt{d} \exp(-(d-1)n + dn \ln(dn) - n \ln n) \\ &= \sqrt{d} \exp(-(d-1)n + (d-1)n \ln(n) + dn \ln d) \end{aligned} \tag{3.4}$$

therefore, as $\sum_{j=0}^{p-1} a_j = n$:

$$\begin{aligned} &\binom{n}{a_0, \cdots, a_{p-1}} \binom{dn}{da_0, \cdots, da_{p-1}}^{-1} \\ =&(1 + o(1)) d^{p/2} \frac{n!}{(dn)!} \exp\left(\sum_{j=0}^{p-1} -(d-1)a_j + (d-1)a_j \ln(a_j) + da_j \ln d\right) \\ =&(1 + o(1)) d^{(p-1)/2} \exp\left((d-1) \sum_{j=0}^{p-1} a_j \ln(a_j) - (d-1)n \ln(n)\right) \end{aligned}$$

We denote $\mathfrak{n}_j = a_j/n$, then :

$$\sum_{j=0}^{p-1} a_j \ln(a_j) = n \ln(n) + n \sum_{j=0}^{p-1} \mathfrak{n}_j \ln(\mathfrak{n}_j)$$

Thus :

$$\begin{aligned} &\binom{n}{a_0, \cdots, a_{p-1}} \binom{dn}{da_0, \cdots, da_{p-1}}^{-1} p^{(d-1)n} \\ =&(1 + o(1)) d^{(p-1)/2} \exp\left((d-1)n \left(\sum_{j=0}^{p-1} \mathfrak{n}_j \ln(\mathfrak{n}_j) + \ln(p)\right)\right) \\ =&(1 + o(1)) d^{(p-1)/2} \exp\left((d-1)n \sum_{j=0}^{p-1} (\mathfrak{n}_j \ln(\mathfrak{n}_j) - 1/p \ln(1/p))\right) \end{aligned} \tag{3.5}$$

Let $H : \ker \operatorname{tr} \to \mathbb{R}$, be such as :

$$H(\boldsymbol{x}) = \sum_{j=0}^{p-1} (x_j + 1/p) \ln(x_j + 1/p)$$

Then :

$$DH(\mathbf{0}).\boldsymbol{h} = \sum_{j=0}^{p-1}(\ln(0+1/p)+1)h_j = (\ln(1/p)+1)\mathrm{tr}(\boldsymbol{h}) = 0 \tag{3.6}$$

We can derive (3.6) also by the fact that the uniform measure on $\{1,\cdots,p\}$ maximises the entropy $-H$. We can then establish the hessian of H at 0 :

$$D^2H(0).\boldsymbol{h}^2 = \frac{1}{1/p+0}\sum_{j=0}^{p-1} h_j^2 = p\|\boldsymbol{h}\|^2$$

thus :

$$\begin{aligned}\sum_{j=0}^{p-1}(\mathfrak{n}_j\ln(\mathfrak{n}_j) - 1/p\ln(1/p)) &= H(\mathfrak{n}-1/p) - H(0) \\ &= |H(\mathfrak{n}-1/p) - H(0)| \\ &= p/2\|\mathfrak{n}-1/p\|^2 + o(\|\mathfrak{n}-1/p\|^2)\end{aligned} \tag{3.7}$$

therefore, we can bound uniformly the first term of (3.1) :

$$\begin{aligned}&\binom{n}{a_0,\cdots,a_{p-1}}\binom{dn}{da_0,\cdots,da_{p-1}}^{-1} p^{(d-1)n} \\ =&(1+o(1))d^{(p-1)/2}\exp\left((d-1)n(p/2\|\mathfrak{n}-1/p\|^2+o(\delta))\right)\end{aligned} \tag{3.8}$$

By the inverse Fourier formula for discrete variables :

$$\mathbb{P}(S_n = d\boldsymbol{a}) = \frac{1}{(2\pi)^p}\int_{2\pi(\mathbb{R}/\mathbb{Z})^p}\phi^n_{X-\mu}(\boldsymbol{t})e^{-i<\boldsymbol{t},d\boldsymbol{a}-n\mu>}d\boldsymbol{t}$$

Outside of the domains $B_j(\epsilon)$ from Proposition 2.2, ϕ is exponentially small :

$$\frac{1}{(2\pi)^p}\int_{2\pi(\mathbb{R}/\mathbb{Z})^p}(1-c(d)\epsilon^2/p^3)^n d\boldsymbol{t} \le e^{-c(d)p\epsilon^2 n} \tag{3.9}$$

Furthermore, the integrand is translation invariant by α_p. Indeed :

$$\forall\lambda\in\mathbb{R},\ \ \exp(i<\boldsymbol{t}+\lambda\alpha_p, X>) = \exp(i<\boldsymbol{t},X>)$$

And $B_j = j\alpha_p + B$, therefore we have :

$$\begin{aligned}\mathbb{P}(S_n = d\boldsymbol{a}) &\le e^{-c(d)\epsilon^2 n/p^3} + \frac{1}{(2\pi)^p}\sum_{j=0}^{p-1}\int_{B_j(\epsilon)}\phi^n_{X-\mu}(\boldsymbol{t})e^{-i<\boldsymbol{t},d\boldsymbol{a}-n\mu>}d\boldsymbol{t} \\ &\le \frac{p}{(2\pi)^p}\int_{B(0,\epsilon)}\phi^n_{X-\mu}(\boldsymbol{t})e^{-i<\boldsymbol{t},d\boldsymbol{a}-n\mu>}d\boldsymbol{t} + e^{-c(d)p\epsilon^2 n}\end{aligned} \tag{3.10}$$

We take, the orthogonal matrix O given by the spectral theorem, such that $O^t\Sigma O = d/pI_{p-1}$. In particular $B(0,\epsilon) = O(B(0,\epsilon))$. The Taylor expansion of the characteristic function is :

$$\begin{aligned}\phi_{X-\mu}(O\boldsymbol{x}) &= \mathbb{E}[1 + i\langle Ox, X-\mu\rangle - \frac{1}{2}\langle Ox, X-\mu\rangle^2 - \frac{i}{6}\langle Ox, X-\mu\rangle^3] + \mathcal{O}(\|x\|^3/p)\\ &= 1 - \frac{1}{2}(O\boldsymbol{x})^t\Sigma(O\boldsymbol{x}) + \mathcal{O}(\|x\|^3/p)\\ &= 1 - \frac{d}{2p}\|x\|^2 + \mathcal{O}(\|x\|^3/p)\end{aligned}$$

Then, if $\epsilon \to 0$:

$$\phi_{X-\mu}(O\boldsymbol{x})^n = (1+o(1))e^{-\frac{dn}{2p}\|x\|^2} \tag{3.11}$$

therefore:

$$\begin{aligned}&\frac{p}{(2\pi)^p}\int_{B(0,\epsilon)} \phi^n_{X-\mu}(\boldsymbol{t})e^{-i<\boldsymbol{t},d\boldsymbol{a}-n\mu>}d\boldsymbol{t}\\ =&\frac{p}{(2\pi)^p}(1+o(1))\int_{B(0,\epsilon)} e^{-\frac{dn}{2p}\|x\|^2}e^{-i\langle x, O^t(d\boldsymbol{a}-n\mu)\rangle}dx\\ \leq&\frac{p}{(2\pi)^p}(1+o(1))\underbrace{\int_{\mathbb{R}^p} e^{-\frac{dn}{2p}\|x\|^2}e^{-i\langle x, O^t(d\boldsymbol{a}-n\mu)\rangle}dx}_{=\text{ Fourier transform of }\mathcal{N}(0,p/dnI_p)}\\ =&(1+o(1))\frac{p\sqrt{\det p/dnI_p}}{(2\pi)^{p/2}}\exp\left(-\frac{p}{2dn}\|O^t(d\boldsymbol{a}-n\mu)\|^2\right)\\ =&(1+o(1))p\left(\frac{p}{dn2\pi}\right)^{p/2}\exp\left(-\frac{p}{2dn}\|d\boldsymbol{a}-n\mu\|^2\right)\\ =&(1+o(1))p\left(\frac{p}{dn2\pi}\right)^{p/2}\exp\left(-\frac{pdn}{2}\|\mathfrak{n}-1/p\|^2\right)\end{aligned}$$

This last term cancels the exponent in (3.8), combining this with (3.10) we have for any ϵ, d-small enough :

$$\begin{aligned}&\frac{1}{|M_{d,p}|}\sum_{\boldsymbol{v}\in\mathcal{S}^n(a_0,\cdots,a_{p-1})} |\{\mathcal{G}\in M_{n,d}|A(\mathcal{G})\boldsymbol{v}=0\}\\ \leq&\left(1+\underset{\epsilon\to 0}{o(1)}\right)p\left(\frac{p}{n2\pi}\right)^{p/2}\exp(-pn/2\|\mathfrak{n}-1/p\|^2)\\ &+\mathcal{O}(\exp((d-1)pn/2\|\mathfrak{n}-1/p\|^2-p\epsilon^2 n))\end{aligned} \tag{3.12}$$

We have $|\mathcal{E}| \leq |\mathbb{F}_p^n| = n^p$ so by taking, $\epsilon \gg \delta$, and $\epsilon^2 \gg \ln n/n$, the last term is small, uniformly on p :

$$\begin{aligned}&\sum_{\boldsymbol{a}\in\mathcal{E}}\mathcal{O}(\exp((d-1)pn/2\|\mathfrak{n}-1/p\|^2-p\epsilon^2 n))\\ =&\mathcal{O}(\exp(pn[(d-1)/2\delta^2-\epsilon^2+\ln(n)/n])\\ =&o(1)\end{aligned} \tag{3.13}$$

We have seen in Section 2.3, if $\sum_{j=0}^{p-1} ja_j \not\equiv 0 \mod p$ then $\mathbb{P}(S_n = d\boldsymbol{a}) = 0$. However we can replace the terms by an average over $\mathcal{E}$:

$$p\exp(-pn/2\|\mathfrak{n}-1/p\|^2) = (1+o(1))\sum_{j=0}^{p-1}\exp(-pn/2\|\mathfrak{n}+(e_j-e_0)/n-1/p\|^2)$$

therefore, we replace the sum over $\sum_{j=0}^{p-1} ja_j \equiv 0 \mod p$, with the sum over all $\mathcal{E}$, gaining a factor $1/p$. Finally, the set of points $\mathfrak{n} - 1/p \in \mathcal{E}$ is a subset of the lattice $\mathbb{Z}/n$. The volume of the fundamental domain is $1/n^p$, therefore :

$$\begin{aligned}\sum_{a\in\mathcal{E}} \left(\frac{p}{n2\pi}\right)^{p/2} \exp(-pn/2\|\mathfrak{n} - 1/p\|^2) \leq \left(\frac{np}{2\pi}\right)^{p/2} \int_{B(0,\delta)} e^{-pn/2\|x\|^2} dx \\ \leq 1\end{aligned} \tag{3.14}$$

The Proposition follows by combining (3.12), (3.13) and (3.14) for $\epsilon \gg \delta$, and $\epsilon^2 \gg \ln n/n$. □

3.2 Large deviation estimate

In this section, we show that the sum of terms in (3.1) corresponding to non-equidistributed p-tuples, is small.

Proposition 3.2. *Let $d \geq 3$ be a fixed integer, p a prime such that $\gcd(p, d) = 1$, with $p \ll n$. We have :*

$$\sum_{\boldsymbol{a}\in\mathcal{N}} \sum_{\boldsymbol{v}\in\mathcal{S}^n(a_0,\cdots,a_{p-1})} |\{\mathcal{G}|A(\mathcal{G})v = 0\}| = o(1)|M_{n,d}| \tag{3.15}$$

Thanks to our work on (3.5), we know that :

$$\begin{aligned}&\binom{n}{a_0, \cdots, a_{p-1}} \binom{dn}{da_0, \cdots, da_{p-1}}^{-1} p^{(d-1)n} \\ =& e^{\mathcal{O}(p)} \exp\left((d-1)n\left(\sum_{j=0}^{p-1} \mathfrak{n}_j \ln(\mathfrak{n}_j) + \ln(p)\right)\right)\end{aligned}$$

We can estimate the last term with a Chernoff bound. For any $t \in \mathbb{R}^p$:

$$\begin{aligned}\mathbb{P}(S_n = d\boldsymbol{a}) &\leq \mathbb{P}(\exp(< \boldsymbol{t}, S_n > - d < \boldsymbol{t}, \boldsymbol{a} >) = 1) \\ &\leq \mathbb{E}(e^{<\boldsymbol{t},X>})^n e^{-d<\boldsymbol{t},\boldsymbol{a}>}\end{aligned}$$

thus :

$$\begin{aligned}\mathbb{P}(S_n = d\boldsymbol{a}) &\leq \inf_{\boldsymbol{t}\in\mathbb{R}} \mathbb{E}(e^{<\boldsymbol{t},X>})^n e^{-d<\boldsymbol{t},\boldsymbol{a}>} \\ &= \exp(-d < \boldsymbol{t}, \boldsymbol{a} > + \ln \inf_{\boldsymbol{t}\in\mathbb{R}^p} \mathbb{E}(e^{<\boldsymbol{t},X>})^n) \\ &= \exp(n \inf_{\boldsymbol{t}\in\mathbb{R}^p} \ln \mathbb{E}(e^{<\boldsymbol{t},X>}) - d < \boldsymbol{t}, \boldsymbol{a} >)\end{aligned} \tag{3.16}$$

We define the rate function :

$$I(\mathfrak{n}) = -(d-1)\left(\sum_{j=0}^{p-1} \mathfrak{n}_j \ln(\mathfrak{n}_j) + \ln(p)\right) - \left(\inf_{\boldsymbol{t}\in\mathbb{R}^p} \ln \mathbb{E}(e^{<\boldsymbol{t},X>}) - d < \boldsymbol{t}, \mathfrak{n} >\right)$$

Where :

$$\frac{1}{|M_{n,d}|} \sum_{\boldsymbol{a}\in\mathcal{N}} \sum_{\boldsymbol{v}\in\mathcal{S}^n(a_0,\cdots,a_{p-1})} |\{\mathcal{G}|A(\mathcal{G})v = 0\}| = \sum_{\boldsymbol{a}\in\mathcal{N}} e^{\mathcal{O}(p)} e^{-nI(\mathfrak{n})}$$

Proposition 3.3. *Let $d \geq 3$ be a fixed integer, p a prime such that $\gcd(p, d) = 1$. Then for δ sufficiently small, there exists a constant $c(d)$, such that :*

$$I(\mathfrak{n}) \geq \frac{c(d)\delta^3}{p} \tag{3.17}$$

unless $\max_{0 \leq k \leq p-1} |\mathfrak{n}_k - 1/p| \leq \delta/p$ *or* $\mathfrak{n}_0 \geq 1 - \delta/p$.

Lemma 2. *Let $d \geq \epsilon \geq 0$, and $a_1, \cdots, a_d \in \mathbb{R}_+^d$, be such that :*

$$\frac{\min a_i}{\max a_i} \leq \frac{1}{1+\epsilon}$$

Then :

$$\prod_{k=1}^{d} a_k^{1/d} \leq (1 - (\epsilon/d)^2/2) \sum_{k=1}^{d} \frac{a_k}{d} \tag{3.18}$$

Proof of the Lemma 2. We can assume that a_i are sorted in ascending order, and we denote $\sum a_i/d = m$. We define $b_i = a_i$, if $i \neq 1, d$; and $b_1 = a_1 a_d/m$, $b_d = m$, then :

$$\prod_{k=1}^{d} b_k = \prod_{k=1}^{d} a_k \tag{3.19}$$

We have also :

$$\begin{aligned} b_1 + b_d - (a_1 + a_d) &= m - a_1 - a_d + a_1 a_d/m \\ &= (m - a_1)(m - a_d)/m \\ &\leq (m - a_1)(m - (1+\epsilon)a_1)/m \\ &\leq -\epsilon(m - a_1) \end{aligned} \tag{3.20}$$

We can then bound a_1 :

$$dm = \sum_{k=1}^{d} a_i \geq a_d + (d-1)a_1 \geq (d+\epsilon)a_d \tag{3.21}$$

Finally, combining (3.19), (3.20), (3.21), and the usual AM–GM inequality, we get :

$$\begin{aligned} \prod_{k=1}^{d} a_k^{1/d} &\leq \sum_{k=1}^{d} \frac{b_k}{d} \\ &\leq m + (b_1 + b_d - (a_1 + a_d))/d \\ &\leq m - \epsilon/d(m - a_1) \\ &\leq \left(1 - \left(1 - \frac{d}{d+\epsilon}\right)\epsilon/d\right) m \\ &\leq (1 - (\epsilon/d)^2/2)m \end{aligned}$$

□

Proof of the Proposition 3.3. We $\boldsymbol{t} = (d-1)/d \ln \mathfrak{n} \in \overline{\mathbb{R}}^p$ then all terms except $-(d-1)\ln p - \ln \mathbb{E}$, cancel-out and I is lower bounded by :

$$\begin{aligned}
I(\mathfrak{n}) &\geq -(d-1)\ln(p) - \ln \mathbb{E}(e^{<\boldsymbol{t},X>}) \\
&= -(d-1)\ln(p) + (d-1)\ln(p) - \ln \sum_{\omega\in\mathcal{U}_{d,p}} \prod_{j=0}^{p-1} \mathfrak{n}_j^{\frac{d-1}{d}\omega_j} \\
&= -\ln \sum_{\boldsymbol{a}\in\mathbb{F}^0_{d,p}} \prod_{j=0}^{p-1}\prod_{k=1}^{d} \mathfrak{n}_j^{\frac{d-1}{d}\mathbb{1}_{a_k=j}} \\
&= -\ln \sum_{\boldsymbol{a}\in\mathbb{F}^0_{d,p}} \prod_{k=1}^{d} \mathfrak{n}_{\mathfrak{a}_k}^{\frac{d-1}{d}}
\end{aligned} \tag{3.22}$$

For the d-tuples that meet the conditions of the Lemma 2, we have :

$$\begin{aligned}
\prod_{k=1}^{d} \mathfrak{n}_{\mathfrak{a}_k}^{\frac{d-1}{d}} &= \prod_{i=1}^{d}\prod_{j\neq i} \mathfrak{n}_{\mathfrak{a}_j}^{1/d} \\
&\leq \frac{1-(\epsilon/d)^2/2}{d} \sum_{i=1}^{d}\prod_{j\neq i} \mathfrak{n}_{\mathfrak{a}_j}
\end{aligned}$$

$\mathbb{F}^0_{d,p}$ is stable by permutation, so is the conditions of the Lemma, thus we get :

$$\begin{aligned}
\sum_{\boldsymbol{a}\in\mathbb{F}^0_{d,p}} \prod_{k=1}^{d} \mathfrak{n}_{\mathfrak{a}_k}^{\frac{d-1}{d}} &\leq \frac{1}{d} \sum_{\boldsymbol{a}\in\mathbb{F}^0_{d,p}} \left(1 - \mathbb{1}_{\frac{\min a_i}{\max a_i} \leq \frac{1}{1+\epsilon}} (\epsilon/d)^2/2\right) \sum_{i=1}^{d}\prod_{j\neq i} \mathfrak{n}_{\mathfrak{a}_j} \\
&= \frac{1}{d} \sum_{\boldsymbol{a}\in\mathbb{F}_p^{d-1}} \sum_{i=1}^{d}\prod_{j=1}^{d-1} \mathfrak{n}_{\mathfrak{a}_j} - \frac{\epsilon^2}{2d^3} \sum_{\boldsymbol{a}\in\mathbb{F}'} \sum_{i=1}^{d}\prod_{j=1}^{d-1} \mathfrak{n}_{\mathfrak{a}_j} \\
&= \underbrace{\sum_{\boldsymbol{a}\in\mathbb{F}_p^{d-1}} \prod_{j=1}^{d-1} \mathfrak{n}_{\mathfrak{a}_j}}_{=\text{ sum of products of } d-1 \text{ terms}} - \frac{\epsilon^2}{2d^2} \sum_{\boldsymbol{a}\in\mathbb{F}'} \prod_{j=1}^{d-1} \mathfrak{n}_{\mathfrak{a}_j} \\
&= \left(\sum_{j=0}^{p-1} \mathfrak{n}_j\right)^{d-1} - \frac{\epsilon^2}{d^2} \sum_{\boldsymbol{a}\in\mathbb{F}'} \prod_{j=1}^{d-1} \mathfrak{n}_{\mathfrak{a}_j} \\
&= 1 - \frac{\epsilon^2}{2d^2} \sum_{\boldsymbol{a}\in\mathbb{F}'} \prod_{j=1}^{d-1} \mathfrak{n}_{\mathfrak{a}_j}
\end{aligned}$$

where $\mathbb{F}' \subset \mathbb{F}_p^{d-1}$ is the set of $d-1$ tuples $\boldsymbol{a}$, such that the conditions of the Lemma for $\mathfrak{n}_{\boldsymbol{a}}$ are met.

In the following, we take $\epsilon = \delta/3$ and assume $\epsilon \leq 1/2$. We prove that there exists a constant $c(d)$ that depends only on d, such that :

$$\sum_{\boldsymbol{a}\in\mathbb{F}'} \prod_{j=1}^{d-1} \mathfrak{n}_{\mathfrak{a}_j} \geq \frac{c(d)\delta}{p} \tag{3.23}$$

unless $\max_{0\leq k\leq p-1}|\mathfrak{n}_k - 1/p| \leq \delta/p$ or $\mathfrak{n}_0 \geq 1-\delta/p$.

Without loss of generality, we can assume $\mathfrak{n}$ are sorted in descending order. We take t_1, t_2 the last index such as $\mathfrak{n}_{t_i} \geq \mathfrak{n}_0/(1+\epsilon)^i$. If $\mathfrak{n}_{t_1+1}+\cdots\mathfrak{n}_{p-1} \geq \epsilon$, we can restrict the sum to $a_0 = 0, a_2 \geq t_1$, then :

$$\sum_{\boldsymbol{a}\in\mathbb{F}'}\prod_{j=1}^{d-1}\mathfrak{n}_{\mathfrak{a}_j} \geq n_0(n_{t_1+1}+\cdots\mathfrak{n}_{p-1})\underbrace{\sum_{\boldsymbol{a}\in\mathbb{F}_p^{d-3}}\prod_{j=3}^{d-1}\mathfrak{n}_{a_j}}_{=1^{d-3}} \geq \frac{\epsilon}{p} \tag{3.24}$$

The claim (3.23) follows, so we can assume $\mathfrak{n}_{t_1+1}+\cdots\mathfrak{n}_{p-1} \leq \epsilon$. If $\mathfrak{n}_{t_2}+\cdots\mathfrak{n}_{p-1} \geq \epsilon/p$, by restricting the sum (3.23) over $a_0 \in \{0,\cdots,t_1\}$ and $a_2 \in \{t_2+1,\cdots,p-1\}$ we have :

$$\sum_{\boldsymbol{a}\in\mathbb{F}'}\prod_{j=1}^{d-1}\mathfrak{n}_{\mathfrak{a}_j} \geq (n_0+\cdots\mathfrak{n}_{t_1})(n_{t_2+1}+\cdots\mathfrak{n}_{p-1}) \geq \frac{\epsilon(1-\epsilon)}{p} \geq \frac{\epsilon}{2p} \tag{3.25}$$

We assume now additionally that $\mathfrak{n}_{t_2}+\cdots\mathfrak{n}_{p-1} \leq \epsilon/p$ and consider three cases :

- $t_2 = p-1$ then $\mathfrak{n}_0/\mathfrak{n}_{p-1} < (1+\epsilon)^2 \leq 1+\delta$ and $\max_{0\leq k\leq p-1}|\mathfrak{n}_k - 1/p| \leq \delta/p$.
- $t_2 = 0$ then, if before rearranging $\mathfrak{n}_0$ is the max, then $\mathfrak{n}_0 \geq 1-\epsilon/p \geq 1-\delta/p$. Else, there exists $i \neq 0$, such as $\mathfrak{n}_i \geq 1-\delta/\epsilon$. By restricting the sum to $a_0 = \cdots = a_{d-2} = i$, we get :

$$\sum_{\boldsymbol{a}\in\mathbb{F}'}\prod_{j=1}^{d-1}\mathfrak{n}_{\mathfrak{a}_j} \geq \mathfrak{n}_i^{d-1} \geq (1-\epsilon/p)^{d-1} \geq \frac{1}{2^{d-1}} \tag{3.26}$$

- Else, we have $n_0 \geq \cdots \geq n_{t_2} \geq n_0/(1+\epsilon)^2$, and $t_2 n_0 \geq 1-\epsilon/p$. We will restrict the sum (3.23) over $a_0,\cdots,a_{d-2} \in \{0,\cdots,t-2\}$, and $a_{d-1} \in \{t_2+1,\cdots,p-1\}$. We take q such as :

$$t_2 q \equiv -2(0+1+\cdots+t_2) = -t_2(t_2+1) \mod p \tag{3.27}$$

The number of $d-2$ tuples in $\{0,\cdots,t_2\}$ such as $a_0+\cdots+a_{d-3} \not\equiv q \mod p$ is at least $(t_2-1)t_2^{d-3}$. And for such a $d-2$ tuple $\boldsymbol{a}$ there exists at least one $a_{d-2} \in \{0,\cdots,t_2\}$ such as $a_1+\cdots+a_{d-2} \not\equiv -0,\cdots,-t_2 \mod p$. Otherwise :

$$t_2(a_1+\cdots+a_{d-2})+0+\cdots+t_2 \equiv -(0+\cdots+t_2) \mod p \tag{3.28}$$

and $a_1+\cdots+a_{d-2} \equiv q \mod p$, which leads to a contradiction. Therefore there are least $(t_2-1)t_2^{d-3}$ term in the restriction we made :

$$\sum_{\boldsymbol{a}\in\mathbb{F}'}\prod_{j=1}^{d-1}\mathfrak{n}_{\mathfrak{a}_j} \geq (t_2-1)t_2^{d-3}\left(\frac{1-\epsilon/p}{(1+\epsilon)^2 t_2}\right)^{d-1} \geq \frac{1}{2^{3d}p} \tag{3.29}$$

Combining, (3.24), (3.25), (3.26), (3.27) and (3.29), we can take $c(d) = 1/2^{3d+1}d^2$, and use $-\ln(1-x) \geq x$ to prove the Proposition 3.3. □

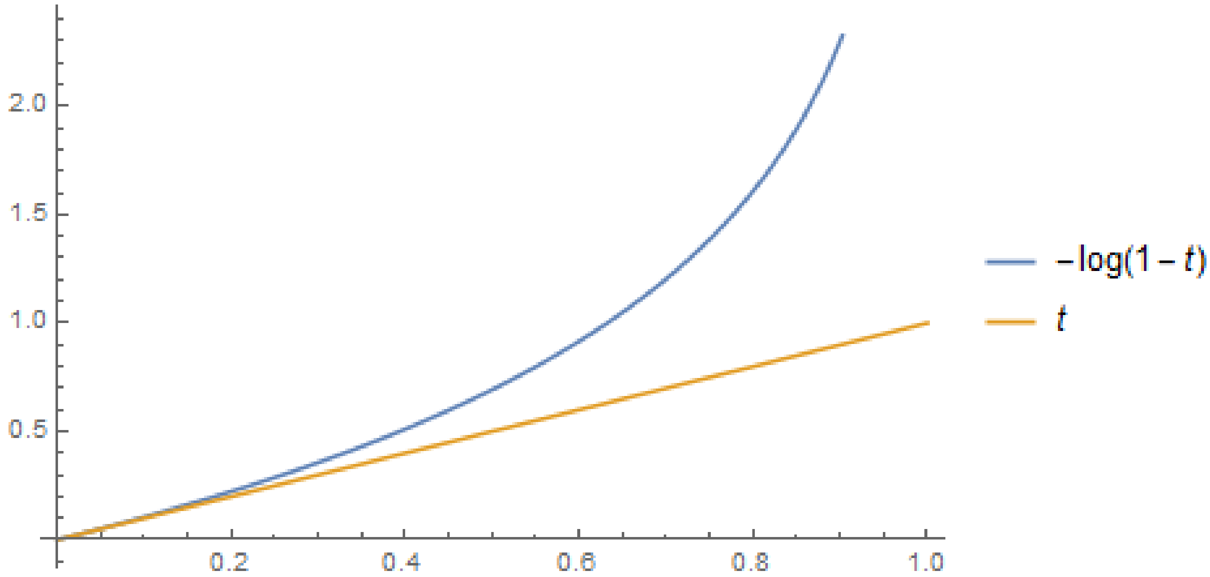

Figure 3: Lower bound of $-\ln(1-t)$

We can then take $\sqrt{\delta}$ from (3.2), and apply Proposition 3.3, to tuples in $\mathcal{N}$ such that $\mathfrak{n}_0 \leq 1-\sqrt{\delta}/p$, and we get :

$$\begin{aligned}&\frac{1}{|M_{n,d}|}\sum_{\substack{\boldsymbol{a}\in\mathcal{N}\\ a_0\leq n(1-\sqrt{\delta}/p)}}\sum_{\boldsymbol{v}\in\mathcal{S}^n(a_0,\cdots,a_{p-1})}|\{\mathcal{G}|A(\mathcal{G})v=0\}|\\ &=\sum_{\boldsymbol{a}\in\mathcal{N}} e^{\mathcal{O}(p)}e^{-nc(d)\delta^{3/2}/p}=\mathcal{O}(\exp(-nc(d)\delta^{3/2}/p+\ln(n)p))\end{aligned}$$

We can take any $\omega_n \to +\infty$, with $\omega_n \ll (n/\ln(n))^{2/3}$. Then with $\delta = \omega_n(\ln(n)/n)^{2/3}$ and $p \ll \omega_n^3$, such that this sum converges to 0. So we only need to prove Proposition 3.2, for $a_0 \geq n(1-\sqrt{\delta}/p)$.

Proof of Proposition 3.2. We have $a_0 = n-m$, $2 \leq m \leq n\sqrt{\delta}/p$ (if we recall that $\boldsymbol{a} \in \Phi(\mathbb{F}_{d,p}^0)$, $m=1$ is impossible). We re-estimate the first factor of the sum on random walks (3.1) :

$$\begin{aligned}\binom{n}{a_0,\cdots,a_{p-1}}\binom{dn}{da_0,\cdots,da_{p-1}}^{-1} &\leq \frac{n^m}{(d(n-m))^{dm}}\prod_{j=1}^{p-1}\frac{(da_j)!}{a_j!}\\ &\leq \frac{e^{\mathcal{O}(m)}}{n^{(d-1)m}}\prod_{j=1}^{p-1}\frac{(da_j)!}{a_j!}\end{aligned}\tag{3.30}$$

If $\omega_1+\cdots+\omega_d = d\boldsymbol{a}$ there is at most the choice of dn_j times the number j in a vector of length dn, that is to say the multinomial:

$$\begin{aligned}&\left|\left\{(\boldsymbol{u}_k)_{k\leq n}\in\mathcal{U}_{d,p}^n|\sum_{k=1}^n\mathbf{u}_k=d(a_0,\cdots,a_{p-1})\right\}\right|\\ &\leq\frac{(dm)!}{(da_1)!\cdots(da_{p-1})!}\left|\left\{(\boldsymbol{u}_k)_{k\leq n}\in\mathcal{U}_{d,p}^n|\left(\sum_{k=1}^n\mathbf{u}_k\right)_0=da_0\right\}\right|\end{aligned}$$

Furthermore $da_1 + \cdots + da_j = dm$ and if $\boldsymbol{u}_k \neq (d, 0 \cdots)$, then $\sum_{i=1}^{p-1} (\boldsymbol{u}_k)_i \geq 2$, so there is at most $dm/2$ vector $\boldsymbol{u}_k$ not equal to $(d, 0, \cdots)$, so :

$$\begin{aligned} &\left| \left\{ (\boldsymbol{u}_k)_{k \leq n} \in \mathcal{U}_{d,p}^n \middle| \sum_{k=1}^{n} \mathbf{u}_k = d(a_0, \cdots, a_{p-1}) \right\} \right| \\ \leq &\frac{(dm)!}{(da_1)! \cdots (da_{p-1})!} \binom{n}{dm/2} \\ \leq &\frac{(dm)!}{(da_1)! \cdots (da_{p-1})!} \frac{n^{dm/2}}{(dm/2)!} \end{aligned} \tag{3.31}$$

Putting (3.30) and (3.31) together, the total contribution of the terms in (3.15) is bounded by :

$$\begin{aligned} \sum_{m=2}^{n\sqrt{\delta}/p} \sum_{a_1 + \cdots a_{p-1} = m} \frac{e^{\mathcal{O}(m)} (dm)!}{a_1! \cdots a_{p-1}!} \frac{1}{(dm/2)! n^{(d/2-1)m}} &\leq \sum_{m=2}^{n\sqrt{\delta}/p} \frac{e^{\mathcal{O}(m)} (dm)! p^m}{m! (dm/2)! n^{(d/2-1)m}} \\ &\leq \sum_{m=2}^{n\sqrt{\delta}/p} \left(\frac{\mathcal{O}(1) m^{d/2} p}{m n^{d/2-1}} \right)^m \end{aligned}$$

We can take for example $\omega_n = n^{1/3}$, such that $\sqrt{\delta} = \ln n^{1/3} / n^{1/6}$, then :

$$\begin{aligned} \sum_{m=2}^{n\sqrt{\delta}/p} \left(\frac{\mathcal{O}(1) m^{d/2} p}{m n^{d/2-1}} \right)^m &\leq n\sqrt{\delta} (\mathcal{O}(1) \sqrt{\delta}^{d/2-1})^{10} + o(1) \\ &= \mathcal{O}(\ln(n) n^{-(5d-9)/6+1}) + o(1) \\ &= \mathcal{O}(\ln(n) n^{-2}) = o(1) \end{aligned} \tag{3.32}$$

□

Combining Proposition 3.1 and Proposition 3.2, we have proven that $|\{(w \neq 0 \in \mathbb{F}_p^n, G \in M_{n,d}) | A(G)w = 0\}| \sim |M_{n,d}|$, for $p \ll \omega_n^3 = n$; which proves Theorem 1.1 for directed graphs.

Part II
Probability of having a certain eigenvalue

Fix $\lambda \in \mathfrak{A}$ the set of algebraic integers. Let P be its minimal monic polynomial and h be its degree.

If we analyze the preliminary work done for the proof of Theorem 1.1, we see that we introduced a coloring $f_{\mathcal{P}}$ on half edges or fibers such that the edges $i \to j$ is colored w_j and $(A(G)w)_i$ is the sum of the colors of the fibers starting from i. We then counted how many graph lead to the same coloring, and sum on the number of coloring to get all the graphs in $M_{n,d}$. Finally we got a condition of the coloring ($\Phi(f(k')) \in \mathcal{U}_{d,p}$) to count the number of graphs such that $A(G)w = 0$. However for other eigenvalues the condition $P(A(G))w = 0$ involves information not only on the vertices adjacent to i but its entire h-neighborhood, therefore to generalize the process, we need to create a much more precise coloring that encodes the entire h-neighborhood of a point.

Once we have this coloring we need to determine a condition similar to $\Phi(f(k')) \in \mathcal{U}_{d,p}$. This can be achieved by decomposing P in a specific polynomial base Q_k such that $Q_k(A(G))_{ij}$ gives the number of non backtracking walks from i to j of length k and noticing that this number is the same as the number of apparition of j in depth k of the universal covering of G centered in i.

4 Generalized configuration model and number of graphs with same h-neighborhood

4.1 Colored configuration model

4.1.1 Directed multi-graphs with colors

Let E be a finite set, with an arbitrary total order. Each pair $(i, j) \in \mathcal{C} = E^2$ is interpreted as a color. Define the subsets of colors :

$$\mathcal{C}_< = \{(i, j) \in \mathcal{C} |\ i < j\}$$

and we define $\mathcal{C}_=, \mathcal{C}_\leq, \mathcal{C}_\neq, \mathcal{C}_>$ in the obvious way, and $\overline{(i, j)} = (j, i)$ the conjugate color.

We consider $\widehat{\mathcal{G}}(\mathcal{C})$ the class of $\mathcal{C}$-colored directed multi-graphs. $G \in \widehat{\mathcal{G}}(\mathcal{C})$ if $G = (V, \omega)$, where $V = [\![1, n]\!]$ and $\omega = \{\omega_c\}_{c\in\mathcal{C}}$, where $\omega_c : V^2 \to \mathbb{N}$ is a map with the following properties :

- $\omega_c(u, v) = \omega_{\bar{c}}(v, u)$.
- if $c \in \mathcal{C}_=$, $\omega_c(u, u)$ is even.

We consider also $\mathcal{G}(\mathcal{C})$ the simple graphs of $\widehat{\mathcal{G}}(\mathcal{C})$. The interpretation is that, for any c $\omega_c(u, v)$ is the number of directed edges of color c from u to v (counted double if $c \in \mathcal{C}_=$). If $G \in \widehat{\mathcal{G}}(\mathcal{C})$, set:

$$D_c(u) = \sum_{v\in V} \omega_c(u, v)$$

$D(u) = \{D_c(u); c \in \mathcal{C}\}$ (can be see as an integer matrix). The vector $\mathbf{D} = \{D(u); u \in V]\}$ will be called the degree sequence of G.

4.1.2 Generalized configuration model

Let $\mathcal{D}_n$ denote the set of n-tuple of non-negative integer matrix $D(i) = \{D_c(i); c \in \mathcal{C}\}$ such that :

$$S = \sum_{i=1}^{n} D(i)$$

is a symmetric matrix with even coefficients on the diagonal, $S = \{S_c; c \in \mathcal{C}\}$. For a $\mathbf{D} \in \mathcal{D}_n$, $\widehat{\mathcal{G}}(\mathbf{D})$ is the set of colored multi-graphs which degree sequence coincides with $\mathbf{D}$. A graph of $\widehat{\mathcal{G}}(\mathbf{D})$ is the result of the superposition of the multi-graphs G_c for $c \in \mathcal{C}_\leq$ with degree sequence D_c.

Configuration model for $c \in \mathcal{C}_=$. When $c \in \mathcal{C}_=$, $\omega_c(u,v) = \omega_c(v,u)$, so G_c is an undirected graph of degree sequence D_c. We may use the usual construction of the undirected configuration model provided in introduction, i.e. uniform pairing of $S_c = \sum D_c(i)$ points.
Let Σ_c be the set of configurations, i.e. pairing of S_c points, and for $\sigma_c \in \Sigma_c$, $\Gamma(\sigma_c)$ be the multi-graph resulting form the pairing σ_c.

Lemma 3. *Fix $c \in \mathcal{C}_=$. Let H be a multi-graph with degree sequence D_c, the number of configuration resulting in H is given by :*

$$n_c(H) = \frac{\prod_{i=1}^{n} D_c(i)!}{\prod_{i=1}^{n} (\omega_c(i,i)/2)! 2^{\omega_c(i,i)/2} \prod_{i<j} \omega_c(i,j)!} \tag{4.1}$$

Proof. We need to count the number of matchings σ_c that pair $\omega_c(i,j)$ elements of sets of cardinal $D_c(i)$ and $D_c(j)$ (if $i \neq j$), and the number of $\omega_c(i,i)/2$ internal pairings of a set of cardinal $D_c(i)$.

- For the first category, once we choose $\omega_c(i,j)$ elements on each side we have $\omega_c(i,j)!$ matchings that produce the same graph
- For the second category, once we choose $\omega_c(i,i)$ elements, we have $\omega_c(i,i)!!$ pairings that produce the same graph.

Finally for each node i, the number of ways of choosing these elements to pair is exactly the multinomial :

$$\binom{D_c(i)}{\omega_c(i,1)\cdots\omega_c(i,n)} = \frac{D_c(i)!}{\omega_c(i,1)!\cdots\omega_c(i,n)!}$$

Putting all together the total number of configuration resulting in H is :

$$\begin{aligned}
&\prod_{i=1}^{n} \binom{D_c(i)}{\omega_c(i,1)\cdots\omega_c(i,n)} \prod_{i=1}^{n} \frac{\omega_c(i,i)!}{(\omega_c(i,i)/2)! 2^{\omega_c(i,i)/2}} \prod_{i>j} \omega_c(i,j)! \\
&= \prod_{i=1}^{n} D_c(i)! \frac{\prod_{i\geq j} \omega_c(i,j)!}{\prod_{i,j\leq n} \omega_c(i,j)!} \frac{1}{\prod_{i=1}^{n} (\omega_c(i,i)/2)! 2^{\omega_c(i,i)/2}} \\
&= n_c(H)
\end{aligned}$$

□

Configuration model for $c \in \mathcal{C}_<$. We have $\omega_c(u,v) = \omega_{\overline{c}}(v,u)$, $D_c(i)$ represents the number of outgoing/incoming edges at the node i. We consider the bipartite version of the previous construction i.e. pairing $S_c = \sum D_c(i)$ points with another S_c points.
Let Σ_c be the set of configurations, i.e. permutations of a set of S_c points, and for $\sigma_c \in \Sigma_c$, $\Gamma(\sigma_c)$ be the directed multi-graph resulting from the matching σ_c and $\sigma_{\overline{c}} = \sigma_c^{-1}$.

Lemma 4. *Fix $c \in \mathcal{C}_<$. Let H be a directed multi-graph with degree sequence D_c, the number of configuration resulting in H is given by :*

$$n_c(H) = \frac{\prod_{i=1}^{n} D_c(i)! D_{\bar{c}}(i)!}{\prod_{i,j \leq n} \omega_c(i,j)!} \tag{4.2}$$

Proof. We have to count the number of bijective maps σ_c that map $\omega_c(i,j)$ elements of a set of cardinality $D_c(i)$ to a set of cardinality $D_{\bar{c}}(j)$. We begin for each i, choosing these subsets of outgoing and incoming edges, this can be done :

$$\binom{D_c(i)}{\omega_c(i,1)\cdots\omega_c(i,n)}\binom{D_{\bar{c}}(i)}{\omega_{\bar{c}}(i,1)\cdots\omega_{\bar{c}}(i,n)}$$

ways. Then, for each of these subsets there are $\omega_c(i,j)!$ distinct bijections that produce the same graph. Therefore the total number of configuration resulting in H is :

$$\prod_{i=1}^{n}\binom{D_c(i)}{\omega_c(i,1)\cdots\omega_c(i,n)}\binom{D_{\bar{c}}(i)}{\omega_{\bar{c}}(i,1)\cdots\omega_{\bar{c}}(i,n)}\prod_{i,j\leq n}\underbrace{\omega_c(i,j)!}_{\omega_{\bar{c}}(j,i)!}$$
$$= n_c(H)$$

□

Generalized configuration model We now put together these colored graphs, let $\Sigma = \{\Sigma_c; c \in \mathcal{C}_\leq\}$ the set of configurations. The map $\Gamma : \Sigma \to \widehat{\mathcal{G}}(\mathbf{D})$ is the superposition of the $\Gamma(\sigma_c)$ defined above. The configuration model, denoted $CM(\mathbf{D})$, is the law of $\Gamma(\sigma)$, when σ is chosen uniformly over Σ.

Lemma 5. *Let $\mathbf{D} \in \mathcal{D}_n$, G with distribution $CM(\mathbf{D})$ and $H \in \widehat{\mathcal{G}}(\mathbf{D})$. We have:*

$$\mathbb{P}(G = H) = \frac{1}{b(H)} \frac{\prod_{c\in\mathcal{C}}\prod_{i=1}^{n} D_c(i)!}{\prod_{c\in\mathcal{C}_<} S_c! \prod_{c\in\mathcal{C}_=} (S_c - 1)!!} \tag{4.3}$$

where $S_c = \sum D_c(i)$, and $b(H)$ is defined by:

$$b(H) = \left(\prod_{c\in\mathcal{C}_=}\prod_{i,j\leq n}\omega_c(i,j)!\right)\left(\prod_{c\in\mathcal{C}_=}\prod_{i=1}^{n}(\omega_c(i,i)/2)!2^{\omega_c(i,i)/2}\prod_{i<j}\omega_c(i,j)!\right)$$

Proof. The proof follows from the Lemma 3 and Lemma 4, and by noticing that the cardinality of Σ is given by:

$$|\Sigma| = \prod_{c\in\mathcal{C}_\leq}|\Sigma_c| = \prod_{c\in\mathcal{C}_=}(S_c - 1)!!\prod_{c\in\mathcal{C}_<} S_c!$$

□

In particular if $|\mathcal{C}| = 1$ and H is a simple graph, $b(H) = 1$. We have proven that $G^*_{n,d}$, $G_{n,d}$ conditioned by simple graphs is the uniform law on simple d-regular graphs.

4.2 Graphs with same universal covering neighborhood

Tree are taken with no particular ordering for their children.

4.2.1 Universal covering

Let G be a connected undirected (multi)graph. The universal covering of G is the infinite unrooted tree T_G where we connect each node to copies of its neighbours, repeating the process infinitely. Given any vertex i of G, its h-depth universal covering neighborhood $[G, i]_h$, is the ball of radius h in T_G around any copy of i. We can consider h-depth universal covering neighborhood of non-connected graphs by taking the h-depth universal covering neighborhood of its connected components. If the h-neighborhood of i is a tree then it coincides with its h-depth universal covering neighborhood.

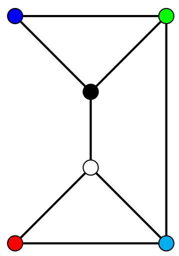

Figure 4: graph G

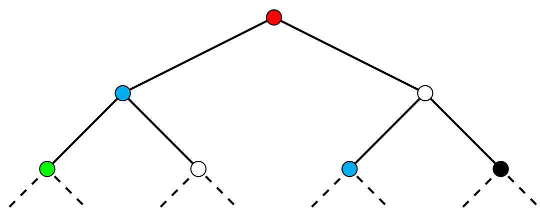

Figure 5: Universal Covering of G

4.2.2 Graph with given fixed depth universal covering neighborhood

Let G be a graph. For h we define the h-depth universal covering neighborhood (abridged h-neighborhood vector) :

$$\psi_h(G) = ([G, 1]_h, \cdots, [G, n]_h)$$

where $[G, i]_h$ is the unlabeled h-depth universal covering neighborhood of i.

We will now describe a procedure which turns a given graph into a directed colored graph. The colors $\mathcal{C}$ are defined as followed. Let $\mathcal{F}$ be the collection of unlabeled neighborhoods $[G(u, v), u]_h$, where $G(u, v)$ is the graph obtained by removing the edge $\leftrightarrow v$. We take $\mathcal{C} = \mathcal{F}^2$. To construct the directed colored graph, for every pair such that u, v is an edge in G, we include a directed edge $u \to v$ with color :

$$(t, t') = ([G(u, v), u]_{h-1}, [G(v, u), v]_{h-1})$$

together with the conjugate edge $v \to u$ colored (t', t). This defines $\tilde{G}$ an element of $\widehat{\mathcal{G}}(\mathcal{C})$, we can also define its degree sequence $\mathbf{D} = \mathbf{D}(\tilde{G})$.

We define the colorblind graph of $H \in \widehat{\mathcal{G}}(\mathcal{C})$, defined by $\overline{H} = (V, \overline{\omega})$, where :

$$\overline{\omega}(u, v) = \sum_{c \in \mathcal{C}} \omega_c(u, v)$$

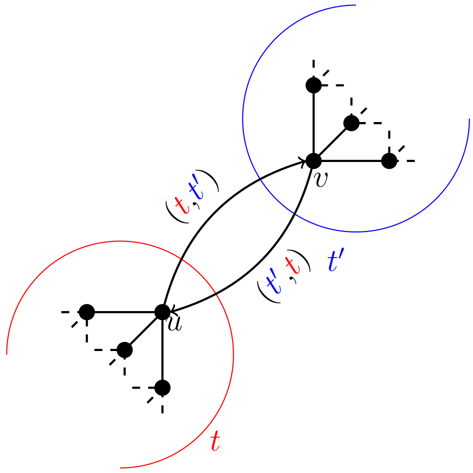

Figure 6: Coloring procedure

Theorem 4.1. *For any* $\Gamma \in \widehat{\mathcal{G}}(\boldsymbol{D})$*, the colorblind graph* $\overline{\Gamma}$ *satisfies* $\psi_h(\overline{\Gamma}) = \psi_h(G)$*, and these are the only ones.*

Proof. Consider first case $h = 1$, if $\Gamma \in \widehat{\mathcal{G}}(\mathbf{D})$, the 1-neighborhood of i in $\overline{\Gamma}$ only depends on $\mathbf{D}(i)$, which is fixed, therefore $\psi_1(\overline{\Gamma}) = \psi_1(G)$.

We now assume that any $\Gamma \in \widehat{\mathcal{G}}(\mathbf{D}(\tilde{G}(h-1)))$ satisfies $\psi_{h-1}(\overline{\Gamma}) = \psi_{h-1}(G)$. For any tree $t \in \mathcal{F}(h)$, we call t_k the k-neighborhood of the root, and t_{k+} the tree of depth $k+1$ obtained from connecting a node to the root of t_k. We denote $t \cup t'$ the tree obtained from fusing the roots of t and t'.
Let $u \to v$ be an edge in Γ with color (t, t'). We note by constraint of $\mathbf{D}$, there must exists in $\tilde{G}$, $\tilde{v}$ and $\tilde{u}$, such as $u \leftrightarrow \tilde{v}$ is colored (t, t') and $\tilde{u} \leftrightarrow v$ is colored (t', t). Therefore, $[G, u]_{h-1} = t \cup t'_{h-2,+}$ and $[G, v]_{h-1} = t' \cup t_{h-2,+}$. By assumption $[\overline{\Gamma}, u]_{h-1} = [G, u]_{h-1}$, therefore the trees $T = [\overline{\Gamma}(u, v), u]_{h-1}$ and $T' = [\overline{\Gamma}(v, u), v]_{h-1}$ must satisfy :

$$T \cup T'_{h-2,+} = t \cup t'_{h-2,+}, \quad T' \cup T_{h-2,+} = t' \cup t_{h-2,+} \tag{4.4}$$

If $T'_{h-2} = t'_{h-2}$ and $T_{h-2} = T_{h-2}$, by (4.4) $T = t, T' = t'$. Truncating at depth $h-2$, we get a similar equation :

$$T_{h-2} \cup T'_{h-3,+} = t_{h-2} \cup t'_{h-3,+}, \quad T'_{h-2} \cup T_{h-3,+} = t'_{h-2} \cup t_{h-3,+}$$

by reiterating this reasoning we see that, $T = t, T' = t'$ iff $T_1 = t'_1$ and $T'_1 = t'_1$, which is guaranteed as G and $\overline{\Gamma}$ have same degree sequence. Therefore we have shown that :

$$[\overline{\Gamma}(u, v), u]_{h-1} = t, \quad [\overline{\Gamma}(v, u), v]_{h-1} = t' \tag{4.5}$$

In particular this shows that Γ is uniquely determined by $\overline{\Gamma}$.

This is true for any v connected to u, therefore :

$$[G, u]_h = \bigcup_{v \neq u} [G(v, u), v]_{h-1} = \bigcup_{\tilde{v} \neq u} [\overline{\Gamma}(\tilde{v}, u), \tilde{v}]_{h-1} = [\overline{\Gamma}, u]_h \tag{4.6}$$

On the contrary if a graph G' has the same h-neighborhood as G, then its degree sequence $\mathbf{D}(\tilde{G}')$, is the same as $\tilde{G}$, there for $\tilde{G}' \in \widehat{\mathcal{G}}(\mathbf{D})$ and $G' = \overline{\tilde{G}'}$

□

Remark. Instead of unlabeled neighborhood we can take colored neighborhoods the proof stays the same.

5 Reconstruction of a graph with its colored universal covering neighborhood

5.1 Erdos–Gallai theorem

We call $\mathbf{d} \in \mathbb{N}^n$ graphic if there exists a simple graph with degree sequence $\mathbf{d}$.

Theorem 5.1 (Erdős-Gallai [6]). *$\mathbf{d} \in \mathbb{N}^n$ is graphic iff its sum is even and after reordering $\mathbf{d}$ in decreasing order, for each integer $k \in [\![1, n]\!]$:*

$$\sum_{i \leq k} d_i \leq k(k-1) + \sum_{i>k} \min(k, d_i) \tag{5.1}$$

Proof. Necessity is immediate as the sum is twice the number of edges of a graph and the right side of the condition is the maximum contribution of the sum of the first k degrees : $k(k-1)$ internal edges (counted twice) and for $i > k$ $\min(d_i, k)$ external edges. For the sufficiency we will give an algorithmic proof given by Tripathi [8]. We assume $\mathbf{d}$ is reordered in decreasing order.

Let a subrealization of $\mathbf{d}$ be a graph with n vertices v_i, such that the degree of v_i is lower then d_i. The initial subrealization has no edges.
Let the critical index r be the first index such that $\deg(v_i) \neq d_i$. Except the trivial case, at first $r = 1$. Let $S = \{v_{r+1}, \cdots, v_n\}$ we assume in our algorithm that they no internal edges in S. We will provide an algorithm that will decrease $|d_r - \deg(v_r)|$ while fixing $\deg(v_i)$ for $i < r$.

- Case 0 : $v_r \not\leftrightarrow v_i$ for some i such that $\deg(v_i) < d_i$ we add $v_r \leftrightarrow v_i$.
- Case 1 : $v_r \not\leftrightarrow v_i$ for some i such that $i < r$, then $\deg(v_i) = d_i > \deg(v_r)$, thus there exists a vertex u adjacent to v_i but not to v_r.
  - if $d_r - \deg(v_r) \geq 2$ we replace $u \leftrightarrow v_i$ by $u \leftrightarrow v_r \leftrightarrow v_i$
  - if $d_r - \deg(v_r) = 1$, since there is an even number of edges (counted double) to be distributed therefore with an argument of parity one $k > r$ must be such that $\deg(v_k) < d_k$, since we have considered Case 0, we have $v_k \leftrightarrow v_r \not\leftrightarrow u \leftrightarrow v_i$ and we can replace this chain by $v_i \leftrightarrow v_k$ and $v_k \leftrightarrow v_r \leftrightarrow u$.
- Case 2 : all $v_i, i < r$ are adjacent to v_r, and $\deg(v_k) \neq \min(r, d_k)$ for some k, since there are no edges internal to S, $\deg(v_k) \leq r$ then we must have $\deg(v_k) < d_k$ and since we have considered Case 0, $v_k \leftrightarrow v_r$. Since $\deg(v_k) < r$ there exists $i < r$ such that $v_i \not\leftrightarrow v_k$, finally using the same argument that in Case 1 we have u adjacent to v_i but not v_r. We can now replace $u \leftrightarrow v_i$ by $u \leftrightarrow v_r$ and $v_i \leftrightarrow v_k$ and we still have $\deg(v_k) \leq d_k$.

- Case 3 : all $v_i, i < r$ are adjacent to v_r, and $v_1, \cdots, v_{r-1}$ is not a complete graph, there is $i \neq j < r$ such that $v_i \not\leftrightarrow v_j$. By the same argument then in Case 1 there exists $u, w \in S$ (possibly equal) such that they are adjacent to v_i, v_j respectively, and not v_r. We can replace $u \leftrightarrow v_i, w \leftrightarrow v_j$ by $v_i \leftrightarrow v_j$ and w or $u \leftrightarrow v$.

If none of these cases apply $V - S = v_1, \cdots, v_r$ is a complete graph and $\deg(v_k) = \min(r, d_k)$ for $k > r$. Since there are no internal edges in S, we have :

$$\sum_{i \leq r} \deg(v_i) = r(r-1) + \sum_{k>r} \min(r, d_k) \tag{5.2}$$

which is impossible since $\sum_{i \leq r} \deg(v_i) < \sum_{i \leq r} d_i$. This concludes the algorithm and the proof. □

One generalization of the problem is the realization of a directed graph :

Theorem 5.2 (Berger [2]). *Let* $\mathbf{d} = (d_i^+, d_i^-) \in (\mathbb{N}^2)^n$. *Then* $\mathbf{d}$ *is the sequence of oriented in/out degrees of a directed graph iff it satisfies the two following conditions:*

$$\sum_{i=1}^{n} d_i^+ = \sum_{i=1}^{n} d_i^- \tag{5.3}$$

and after reordering in lexicographic order, the "directed Erdős-Gallai condition":

$$\sum_{i \leq k} d_i^+ \leq \sum_{i \leq k} \min(d_i^-, k-1) + \sum_{i>k} \min(d_i^-, k) \tag{5.4}$$

5.2 Reconstruction with colors

Fix $w \in \mathbb{F}_p^n$. We color the simple graph $G \in G_{n,d}^*$ by giving the color w_i to i. Let $\mathbb{T}_{p,d}^h$ be the set of rooted trees colored by $\mathbb{F}_p$, of depth h. We take $\mathcal{C} = (\mathbb{T}_{p,d}^h)^2$ and construct $\tilde{G} \in \mathcal{G}(\mathcal{C})$ the same way that in 4.2.2 but with colored trees where we color i with w_i. We still have $\mathbf{D} = \mathbf{D}(\tilde{G})$.

Question Given a degree sequence $\mathbf{D} \in \mathcal{D}_n$, is there a simple graph G associated with $\mathbf{D}$. If that is the case we also say that $\mathbf{D}$ is graphic.
As shown in Bordenave Coste [5] we can reduce this problem to a superposition of graphic sequences and digraphic sequences :

If $\mathbf{D}$ is graphic in particular, for all $c \in \mathcal{C}_=$, $(D_c(i))_{i \in [\![1,n]\!]}$ is graphic and $c \in \mathcal{C}_{\neq}$ the superposition of G_c and G_{c^*} is graphic i.e. $(D_c(i), D_{c^*}(i))_{i \in [\![1,n]\!]}$ is the sequence of oriented in/out degrees of a directed graph.

On the contrary if these two conditions are met we can construct an element of $G \in \widehat{\mathcal{G}}(\mathcal{C})$ that is a superposition of simple directed graphs G_c. As each G_c is loop-less $\overline{G}$ is also loop-less. Finally to prove that $\overline{G}$ is simple, as each G_c is simple, we only need to prove that if there is a edge $u \leftrightarrow v$ in G_c and $G_{c'}$ then $c = c'$. This is exactly the work we have done in Theorem 4.1, where we have shown :

$$c = c' = ([\overline{G}(u,v), u]_{h-1}, [\overline{G}(v,u), v]_{h-1})$$

5.3 Number of simple graphs

Let $\mathcal{D}_n^* \subset \mathcal{D}_n$ be the set of graphical d-regular degree sequence, i.e. by Theorems 5.1, 5.2 the set of $\mathbf{D} = (D_c(i)) \in (\mathbb{N}^{\mathcal{C}})^n$ such that :

- For all $i \in [\![1, n]\!]$:

$$\sum_{c \in \mathcal{C}} D_c(i) = d \tag{5.5}$$

- For $c \in \mathcal{C}_=$, we have after reordering and for any k :

$$2 | \sum_i D_c(i) \tag{5.6}$$

$$\sum_{i \leq k} D_c(i) \leq k(k-1) + \sum_{i > k} \min(k, D_c(i)) \tag{5.7}$$

- For $c \in \mathcal{C}_{\neq}$, we have after reordering and for any k :

$$\sum_i D_c(i) = \sum_i D_{c^*}(i) \tag{5.8}$$

$$\sum_{i \leq k} D_c(i) \leq \sum_{i \leq k} \min(D_{c^*}(i), k-1) + \sum_{i > k} \min(D_{c^*}(i), k) \tag{5.9}$$

For $H \in \mathcal{G}(\mathcal{C})$, $\overline{H}$ is a simple graph, therefore each H_c is simple and $b(H) = 1$ in Lemma 5. Therefore :

$$|\mathcal{G}(\mathbf{D})| = \mathbb{P}(G \in \mathcal{G}(\mathbf{D})) \frac{\prod_{c \in \mathcal{C}_<} S_c! \prod_{c \in \mathcal{C}_=} (S_c - 1)!!}{\prod_{c \in \mathcal{C}} \prod_{i=1}^{n} D_c(i)!} \tag{5.10}$$

With Theorem 4.1 we know that if $\mathbf{D}$ is graphic, $G \in \mathcal{G}(\mathbf{D})$ if and only if all G_c are simple. If we recall the construction of $\widehat{\mathcal{G}}(\mathcal{C})$, these events are independent (if $c \neq c', c'^*$). Therefore, we can reduce the calculation of $\mathbb{P}(G \in \mathcal{G}(\mathbf{D}))$ to the calculation of the probability of the undirected configuration model or the bipartite configuration model having no loops or multi-edges.

$$\mathbb{P}(G \in \mathcal{G}(\mathbf{D}) = \prod_{c \in \mathcal{C}_\leq} \mathbb{P}(G \in \widehat{\mathcal{G}}(\mathbf{D}_c) \text{ is simple}) \tag{5.11}$$

We can then rewrite (5.10) as :

$$\begin{aligned} |\mathcal{G}(\mathbf{D})| = & \prod_{c \in \mathcal{C}_<} p(G \in \widehat{\mathcal{G}}(\mathbf{D}_c) \text{ is simple}) S_c! \prod_{i=1}^{n} \frac{1}{D_c(i)! D_{c^*}(i)!} \\ & \times \prod_{c \in \mathcal{C}_=} p(G \in \widehat{\mathcal{G}}(\mathbf{D}_c) \text{ is simple})(S_c - 1)!! \prod_{i=1}^{n} \frac{1}{D_c(i)!} \end{aligned} \tag{5.12}$$

The probability of being simple will be high is S_c is small, so we will consider the cases were $S_c = \Theta(n)$.

k-cycles of a configuration σ_c for $c \in \mathcal{C}_=$ A k-cycle for a permutation $\sigma_c \in \Sigma_c$, $c \in \mathcal{C}_=$, is a set of k edges $\{e_1, \cdots, e_k\}$ such that for some k distinct fibers $F^c_{v_i}$, e_i joins $F^c_{v_i}$ and $F^c_{v_{i+1}}$ with the convention $F^c_{v_{k+1}} = F^c_{v_1}$.
If we fix a set of edges $\{e_1, \cdots, e_k\}$ there are $2k$ corresponding ways to follow them $(e_i, \cdots, e_{i+k})$ (choosing were to start and which direction to go). If we fix k vertices $v_1, \cdots, v_k$, we have d_{v_i} choices of edges going out and $d_{v_i} - 1$ choices left for edges going in to form a sequence. Therefore the total number of possible k-cycles in σ_c is:

$$\sum_{\substack{J \subset V \\ |J|=k}} \prod_{v \in J} D_c(v)(D_c(v) - 1) \tag{5.13}$$

Finally if we consider k edges out of S_c the probability of them been paired is :

$$\frac{k!}{(S_c - 1) \cdots (S_c - 1 - 2k)} \tag{5.14}$$

Let $X_k(\sigma_c)$ count the number of k-cycles in σ_c. By combining (5.13) and (5.14) we have proven that :

$$\mathbb{E}[X_k] = k! \sum_{\substack{J \subset V \\ |J|=k}} \prod_{v \in J} \frac{D_c(v)(D_c(v) - 1)}{(S_c - 1) \cdots (S_c - 1 - 2k)} \tag{5.15}$$

$k! \sum_{\substack{J \subset V \\ |J|=k}} \prod_{v \in J} D_c(v)(D_c(v)-1)$ is the first terms of the expansion of $(\sum D_c(i)(D_c(i)-1))^k$, indeed :

$$\begin{aligned}
\left(\sum_{i=1}^{n} D_c(i)(D_c(i) - 1)\right)^k &= \sum_{k_1 + \cdots k_n = k} \binom{k}{k_1 \cdots k_n} \prod_{i=1}^{n} (D_c(i)(D_c(i) - 1))^{k_i} \\
&= k! \sum_{\substack{J \subset V \\ |J|=k}} \prod_{v \in J} D_c(v)(D_c(v) - 1) \\
&\quad + \sum_{\substack{k_1 + \cdots k_n = k \\ \exists k_i \notin \{0,1\}}} \binom{k}{k_1 \cdots k_n} \prod_{i=1}^{n} (D_c(i)(D_c(i) - 1))^{k_i}
\end{aligned}$$

As $D_c(i) \leq d$, we have:

$$\begin{aligned}
\sum_{\substack{k_1 + \cdots k_n = k \\ \exists k_i \notin \{0,1\}}} \binom{k}{k_1 \cdots k_n} \prod_{i=1}^{n} (D_c(i)(D_c(i) - 1))^{k_i} &\leq (d(d-1))^k \sum_{\substack{k_1 + \cdots k_n = k \\ \exists k_i \notin \{0,1\}}} \binom{k}{k_1 \cdots k_n} \\
&= (d(d-1))^k \left(n^k - k! \sum_{\substack{J \subset V \\ |J|=k}} \right) \\
&= (d(d-1))^k (n^k - n!/(n-k)!) \\
&= \mathcal{O}(n^{k-1})
\end{aligned}$$

We have taken S_c big so that:

$$\frac{1}{(S_c - 1) \cdots (S_c - 1 - 2k)} = \frac{1}{S_c^k + \mathcal{O}(S_c^{k-1})} = \frac{1}{S_c^k} + \mathcal{O}(n^{-k-1})$$

Let $\lambda_{c,n} = \sum_{i=1}^{n} D_c(i)(D_c(i)-1)/S_c$, we have :

$$\mathbb{E}[X_k] = \lambda_{c,n}^k + \mathcal{O}(n^{-2}) \tag{5.16}$$

See Bollobás [3, Sec 2. Thm 2.16], X_k are asymptotically independent Poisson variables with means $\lambda_{c,n}^k$. Therefore :

$$\begin{aligned} p(G \in \widehat{\mathcal{G}}(\mathbf{D}_c) \text{ is simple}) &= \mathbb{P}(X_1 = 0, X_2 = 0) \\ &\sim \mathbb{P}(X_1 = 0)\mathbb{P}(X_2 = 0) \sim e^{-\lambda_{c,n} - \lambda_{c,n}^2} \end{aligned} \tag{5.17}$$

k-cycles of a configuration σ_c for $c \in \mathcal{C}_{\neq}$ We have the same construction of k-cycles in the directed case except when counting the number of out/in possible edges we have $D_c(v)D_{c^*}(v)$ in place of $D_c(i)(D_c(i)-1)$, therefore :

$$\mathbb{E}[X_k] = k! \sum_{\substack{J \subset V \\ |J|=k}} \prod_{v \in J} \frac{D_c(v)D_{c^*}(v)}{(S_c-1)\cdots(S_c-1-2k)} \tag{5.18}$$

And we also have :

$$p(G \in \widehat{\mathcal{G}}(\mathbf{D}_c) \sim e^{-\lambda_{c,n} - \lambda_{c,n}^2} \tag{5.19}$$

where $\lambda_{c,n} = \sum_{i=1}^{n} D_c(i)D_{c^*}(i)/S_c$

6 Resolution of the equation

6.1 Number of non backtracking path in a regular graph

Let A be the adjacency matrix of a d-regular graph G. Let $\vec{a}_{ij}^l$ count the number of non backtracking walk from i to j of length l. $a_{ik}\vec{a}_{kj}^l$ counts the number of walks from i to j of length $l+1$ starting in k that are non backtracking except maybe the second step. The number of walks from i to j of length $l+1$ that are backtracking only in the second step are walks of form (i,k,ω) where ω is a non backtracking walk of length $l-1$ from i to j and k is different of the first step of ω. As G is d-regular the are exactly $(d-1)\vec{a}_{ij}^{l-1}$ walks of this form. Therefore :

$$\vec{a}_{ij}^{l+1} = \sum_{k=1}^{n} a_{ik}\vec{a}_{kj}^l - (d-1)\vec{a}_{ij}^{l-1} \tag{6.1}$$

Thus we derive :

$$\vec{A}^{l+1} = A\vec{A}^l - (d-1)A^{l-1} \tag{6.2}$$

Let Q_l be a polynomial base defined as follows :

$$\begin{cases} Q_0 &= 1 \\ Q_1 &= X \\ Q_{l+1} &= XQ_l - (d-1)Q_{l-1} \end{cases}$$

Then $\vec{A}^l = Q_l(A)$. These polynomials are actually orthogonal for a certain measure (Kesten-McKay measure) see Alon, Benjamini, Lubetzky & Sodin [1].

Due to the construction of the universal covering a non backtracking walk on it is a simple walk, and the number of non backtracking walk for i to j of length l is exactly the number of times j appears in depth l of the universal covering neighborhood of i.

6.2 Condition on D

We write P in base Q :

$$P = \sum_{l=0}^{h} b_l Q_l \tag{6.3}$$

For a tree $t \in \mathbb{T}_{p,d}^h$ we define $\tilde{P}(t)$ as :

$$\tilde{P}(t) = \sum_{x \in t} b_{d(x)+1} x \tag{6.4}$$

where $d(x)$ is the depth of x. $\tilde{P}$ can by see as a linear function.

Proposition 6.1. *Using the same notations, we have :*

$$(P(A)w)_i = \sum_{t \in \mathbb{T}_{p,d}^h} \sum_{t' \in \mathbb{T}_{p,d}^h} D_{(t',t)}(i)\tilde{P}(t) + b_0 w_i \tag{6.5}$$

Proof. $\sum_{t' \in \mathbb{T}_{p,d}^h} D_{(t',t)}(i)$ gives us the number of times t appears in the h-neighborhood of i. And :

$$(P(A)w)_i = \sum_{l=0}^{h} \sum_{k=1}^{n} \vec{b}_l a_{ik}^l w_k \tag{6.6}$$

The factor $\sum_{k=1}^{n} \vec{a}_{ik}^l b_l w_k$ for $l \geq 1$, counts the number of non backtracking walks starting from i of length l giving them a mass $b_l w_k$ if they end in k. Therefore $\sum_{t' \in \mathbb{T}_{p,d}^h} D_{(t',t)}(i)\tilde{P}(t)$ counts the total contribution of t in $(P(A)w)_i$. □

Therefore the condition on **D**, so that $P(A)w = 0$ is :

$$\forall i \in [\![1, n]\!], \quad \sum_{t \in \mathbb{T}_{p,d}^h} \sum_{t' \in \mathbb{T}_{p,d}^h} D_{(t',t)}(i)\tilde{P}(t) = -b_0 w_i \tag{6.7}$$

If we set $\chi_w(\mathbf{D}) = 1$ if **D** satisfies (6.7) and 0 else, then we have finally:

$$|\{G | P(A(G))w = 0\}| = \sum_{\mathbf{D} \in \mathcal{D}_n^*} \chi_w(\mathbf{D}) \prod_{c \in \mathcal{C}_<} p(G \in \widehat{\mathcal{G}}(\mathbf{D}_c) \text{ is simple}) S_c! \prod_{i=1}^{n} \frac{1}{D_c(i)! D_{c^*}(i)!}$$
$$\times \prod_{c \in \mathcal{C}_=} p(G \in \widehat{\mathcal{G}}(\mathbf{D}_c) \text{ is simple})(S_c - 1)!! \prod_{i=1}^{n} \frac{1}{D_c(i)!}$$

7 Conclusion

We might be able to prove a result similar to Huang [7], $|\{(G,w)|P(A(G))w=0\}|/G^*_{n,d} = \mathcal{O}(1)$ by balancing $p(G \in \widehat{\mathcal{G}}(\mathbf{D}_c)$ is simple) that is exponentially small when S_c is big and $S_c! \prod_{i=1}^n \frac{1}{D_c(i)!D_{c^*}(i)!}$ when S_c is small. This would imply :

$$\forall \lambda \in \mathfrak{A},\ \mathbb{P}(\lambda \in \mathrm{Sp}(G)) = \mathbb{P}(\det P(A(G)) = 0) = o(n^{-\mathfrak{d}}) \tag{7.1}$$

We know that the eigenvalues of a regular graph are in $[-d,d]$. Therefore, for any P irreducible and monic in $\mathbb{Q}$ to have eigenvalues of a d-regular graph as its root means its roots must be in $[-d,d]$. This bounds the size of the coefficient of P, they are at most $\mathcal{O}(d^{k^2})$ of such polynomials with degree lower then k, let $\mathcal{P}_{k,d}$ be the set of such polynomials. Take $h \in \mathbb{R}$, the probability of $G \in G_{n,d}$ to have a eigenvalue of degree lower then h is bounded by:

$$\sum_{P \in \mathcal{P}_{\lfloor h \rfloor,d}} \mathbb{P}(\det P(A(G)) = 0) \leq \exp(\ln(d)h^2 - \mathfrak{d}\ln(n) + \mathcal{O}(1)) \tag{7.2}$$

If we take $h = \mathfrak{c}\sqrt{\ln(n)}$, for $\mathfrak{c}$ d-small enough, then we would have proven that almost surely all eigenvalues of a graph $G \in G_{n,d}$ are of degree greater then $\mathfrak{c}\sqrt{\ln(n)}$. We might get an even better estimate if we determine $|\mathcal{P}_{k,d}|$.

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